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ABS Algorithms for Integer *WZ* Factorization

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ABSTRACT

Classes of integer ABS algorithms have been introduced for solving linear Diophantine equations. The algorithms are powerful methods for developing all matrix factorizations. Here, we provide the conditions for the existence of the integer *WZ* and *ZW* factorizations of an integer matrix. Then, we present algorithms based on the integer *ABS* algorithms for computing the integer *WZ* and *ZW* factorizations of an integer matrix as well as the integer $Z^T XZ$ and $W^T XW$ factorizations of a totally unimodular symmetric positives definite matrix.

Keywords: ABS algorithm, Unimodular matrix, Integer factorization, *WZ* factorization, X factorization.

1. INTRODUCTION

Implicit matrix elimination schemes for the solution of linear systems were introduced by Evans(1993) and Evans and Hatzopoulos (1979). These schemes propose the elimination of two matrix elements simultaneously (as opposed to a single element in Gaussian Elimination) and is eminently suitable for parallel implementation (Evansand Abdullah (1994)).

ABS class of algorithms was constructed for the solution of linear systems $Ax = b$ utilizing some basic ideas such as projection and rank one update techniques (Abaffy and Broyden (1984); Abaffy and Spedicato (1989)).The *ABS* class later extended to solve optimization problems (Abaffy and Spedicato (1989)) and systems of linear Diaphantine equations (see Esmaeili *et al.* (2001); Khorramizadeh and Mahdavi-Amiri (2009); Khorramizadeh and Mahdavi-Amiri (2008)). A scaled version of the linear

ABS class was described in Abaffy and Spedicato (1989). Reviews of *ABS* methods can be found in Spedicato *et al.* (2003) and Spedicato *et al.* (2010).

A basic *ABS* algorithm starts with a nonsingular matrix $H_1 \in R^{n\times n}$ (Spedicato's parameter), as a basis for the null space corresponding to the empty coefficient matrix (no equations). Given the Abaffian matrix H_1 with rows generating the null space of the first $i-1$ equations, the ABS algorithm computes H_{i+1} as a null space generator of the first i equations. Consider the following linear system,

$$Ax = b, x \in R^n, A \in R^{n\times n}, b \in R^n \qquad (1)$$

where *rank(A)* is arbitrary. Obviously, the system (1) is equivalent to the following scaled system,

$$V^T Ax = V^T b, \qquad (2)$$

where *V*, the scale matrix, is an arbitrary nonsingular $n \times n$ matrix.

Let a_i^T be the *i*th row of *A*. A tailored scaled *ABS* algorithm as applied to *A* can be described as follows, where the output variable *r* gives the rank of *A*.

Algorithm 1.The scaled *ABS* (SABS) algorithm.

Step1: Let $H_1 \in R^{n\times n}$ be arbitrary and nonsingular and $v_1 \in R^n$ be an arbitrary nonzero vector. Set $i = 1$ and $r = 0$.

Step2: Compute $s_i = H_i A^T v_i$.

Step3: If $s_i = 0$ then set $H_{i+1} = H_i$ and go to Step 5 (the *i*th row is dependent on the first $i-1$ rows).

Step4: $\{s_i \neq 0\}$ compute $p_i = H_i^T f_i$, where $f_1 \in R^n$ (Broyden's parameter), is an arbitrary vector satisfying $s_i^T f_i \neq 0$ and update H_i by

$$H_{i+1} = H_i - \frac{H_i A^T v_i q_i^T H_i}{q_i^T H_i A^T v_i}, \qquad (3)$$

where $q_i \in R^n$ (Abaffy's parameter) is an arbitrary vector satisfying $s_i^T q_i \neq 0$. Let $r = r + 1$.

Step5: If $i = n$ then Stop (columns of H_{i+1}^T generates the null space of *A*) else define $v_{i+1} \in R^n$, an arbitrary vector linearly independent of $v_i, \ldots, v_i$, let $i = i + 1$ and go to Step 2.

The matrices H_i are generalizations of projections matrices and have been named Abaffians since the First International Conference on *ABS* Methods (Luyoyang (1991)). They probably first appeared in a book by Wedderburn (1934).

An important result of the *ABS* algorithms is the establishment of an implicit matrix factorization

$$V^T AP = L, \tag{4}$$

Where *L* is a lower triangular matrix (see Abaffy and Spedicato (1989)).

Choices of the parameters H_1, v_i, f_i and q_i determine particular methods within the class. The basic *ABS* class is obtained by taking $v_i = e_i$ (Abaffy and Spedicato (1989)), the *i*th unit vector in R^n.

All matrix factorizations can be produced by using the scaled *ABS* algorithm with proper definitions of the parameters (Galantai (2001)).

From (Abaffy and Spedicato (1989)) we recall some properties of the Basic *ABS* algorithms for *LU* factorization.

P1. The implicit *LU* algorithm is defined by the following choices, which are well defined if *A* is regular (all leading principal submatrices are nonsingular)

$$H_1 = I,\ H_{i+1} = H_i - \frac{H_i a_i e_i^T H_i}{e_i^T H_i a_i}, p_i = H_i^T e_i. \tag{5}$$

P2. Let $H_1 = I$, then $\delta_i = e_i^T H_i a_i$ satisfies

$$\delta_1 = a_{1,1}, \ \ \delta_i = \frac{\det(A^{i,i})}{\det(A^{i-1,i-1})}, i > 1, \tag{6}$$

where $A^{i,i}$ is the ith leading principal submatrix of A.

P3. Let the conditions of **P1** be satisfied. Then, the following properties hold:

(a) The first i rows of H_{i+1} are identically zero.

(b) The last $n - i$ column of H_{i+1} is equal to the last $n - i$ column of H_1.

The block *ABS* algorithm, is due to Abaffy and Galantai (1986) for the scaled *ABS* class, and further developed in several papers by Galantai(2001, 2003, 2004), is a block form of the *ABS* algorithm (Abaffy and Spedicato (1989)).

Let A be full rank row and $n_1, ..., n_s$ be positive integer numbers so that $n_1 + ... + n_s = n.$ Assume that nonsingular matrix V is partitioned by $V = [V_1, ..., V_s]$ where $V_i \in R^{n \times n_i}$. The block scale ABS algorithm is as follows.

(1) Determine $F_i \in R^{n \times n_i}$ such that $F_i^T H_i A^T V_i$ is nonsingular and set $P_i = H_i^T F_i$

(2) Update the Abaffian matrix H_i by

$$H_{i+1} = H_i - H_i A^T V_i (Q_i^T H_i A^T V_i)^{-1} Q_i^T H_i, \tag{7}$$

where $Q_i \in R^{n \times n_i}$ is an arbitrary matrix so that $Q_i^T H_i A^T V_i$ is nonsingular.

The remainder of our work is organized as follows. In Section 2, we discuss the integer *ABS* class of algorithms. In Section 3, we present an existence condition for the integer *WZ* factorization. Then, we present an algorithm for computing the integer *WZ* factorization as well as the $Z^T XZ$ factorization of a totally unimodular symmetric positive definite matrix using the block integer *ABS* algorithm. In Section 4, we compute the integer *ZW* factorization by appropriately setting the parameters of the block integer

ABS algorithm. We also compute the integer $W^T XW$ factorization of a totally unimodular symmetric positive definite matrix. An existence condition for the integer *ZW* factorization based on the integer *ABS* algorithm is given. Section 5 illustrates an example for computing the *ZW* factorization. Concluding remarks are given in Section 6.

2. INTEGER *ABS* ALGORITHM

The integer *ABS* (*IABS*) class algorithms for linear Diophantine equations presented by Esmaeili *et al.*(2001) to compute the general integer solution of linear Diophantine equations. Conditions for the existence of an integer solution and determination of all integer solutions of a linear Diophantine system are given in Esmaeili *et al.* (2001).

First we recall some results from number theory and then present the *IABS* algorithm.

Definition 2.1. $A \in R^{n \times n}$ is a unimodular matrix iff $\det(A) = 1$.

If A is unimodular, then A^{-1} is also unimodular.

Definition 2.2.A matrix A is called totally unimodular if each square submatrix of A has determinant equal to 0, +1, or -1. In particular, each entry of a totally unimodular matrix is 0, +1, or -1.

Theorem 2.1.(Fundamental theorem of the single linear Diophantine equation).

Let $a_1,...,a_n$ and b be integer numbers. The Diophantine linear equation $a_1x_1 + ... + a_nx_n = b$ has an integer solution if and only if gcd $(a_1,...,a_n) | b$ (if $n > 1$, then there are an infinite number of integer solutions).

Proof. See Pohst (1993).

The integer *ABS* algorithm (*IABS*) has the following structure, with gcd(u) the greatest common divisor of a vector u.

Algorithm 2.The integer *ABS* (*IABS*) algorithm.

Step1: Let $H_1 \in Z^{n\times n}$ be arbitrary and unimodular matrix. Set $i=1$ and $r=0$.

Step2: Compute $s_1 = H_i A^T v_i$.

Step3: If $s_i = 0$ then set $H_{i+1} = H_i$ and go to Step 5 (the *i*th row is dependent on the first $i=1$ rows).

Step4: $\{s_i \neq 0\}$ compute $\gcd(s_i) = \delta_i$ and $p_i = H_i^T f_i$, where $f_i \in Z^n$ is an arbitrary vector satisfying $s_i^T f_i = \delta_i$ and update H_i by

$$H_{i+1} = H_i - \frac{H_i A^T v_i q_i^T H_i}{q_i^T H_i A^T v_i},$$

where $q_i \in Z^n$ is an arbitrary vector satisfying $s_i^T q_i = \delta_i$. Let $r = r+1$.

Step5: If $i=n$ then stop (columns of H_{i+1}^T generates the null space of *A*) else let $i=i+1$ and go to Step2.

Let $V \in Z^{n\times n}$ be a unimodular matrix. Then, the scaled integer *ABS* algorithm is computed by applying Algorithm 2 on $V^T A$ with $A^T v_i$ replacing a_i.

Theorem 2.2. If all the principal submatrices of *A* are unimodular, the integer *LU* algorithm is well defined.

Proof. See Corollary 4.1 in Zou and Xia (2005).

Corollary 2.1.If *A* is totally unimodular of full rank. Then there exists a row permutation matrix Π so that $\Pi \ A = LU$, where *L* and *U* are integer lower and upper triangular matrix respectively.

Corollary 2.2. Every totally uniomodular symmetric positive definite matrix has an integer *LU* factorization.

Furthermore, in a recent work we have shown that a special version of our approach constructs the Smith normal form of an integer matrix, being utilized in solving linear Diophantine systems of equations (Golpar-Raboky and Mahdavi-Amiri (2012)).

Next, we compute the integer *WZ* and the integer *WZ* factorizations of a non-singular integer matrix as well as the $W^T XW$ and the $Z^T XZ$ factorizations of a totally unimodular symmetric positive definite matrix using the integer *ABS* algorithms.

3. *WZ* FACTORIZATION USING THE BLOCK SCALED *ABS* ALGORITHM

The well known *LU* factorization is one of the most commonly used algorithms to solve linear systems and *WZ* factorization offers an interesting variant of the factorization.

To solve a system of linear equations, the *WZ* factorization procedure proposed in Evans (1993a,b) is convenient for parallel computing. The *WZ* factorization offers a parallel method for solving dense linear systems, where *A* is a square $n \times n$ matrix, and *b* is an *n*-vector.

Definition 3.1. Let s be a real number, and denote by $\lfloor s \rfloor$ $\left(\lceil s \rceil\right)$, the greatest (least) integer less (greater) than or equal to s.

Definition 3.2. We say that a matrix *A* is factorized in an integer *WZ* (*IWZ*) form if

$$A = WZ, \tag{8}$$

where the *W*-matrix and the *Z*-matrix are integer matrices having following structures:

$$W = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}, Z = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \tag{9}$$

with the empty bullets standing for zero and the other bullets standing for possible integer nonzeros.

Definition 3.3. We define an *X*-matrix as follows:

$$X = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \circ & \bullet & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \bullet & \circ \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}. \tag{10}$$

The following theorems express the conditions for the existence of an integer *WZ* factorization of a unimodular matrix (see Rao (1997)). Later, we give a new set of conditions useful for our purposes.

Theorem 3.1. (Factorization Theorem) Let $A \in Z^{n \times n}$ be unimodular. Then *A* has an integer *WZ* factorization if and only if for every *k*, $k = 1, \ldots s$, with $s = \left\lfloor \frac{n}{2} \right\rfloor$, if n is even and $s = \left\lceil \frac{n}{2} \right\rceil$, if *n* is odd, the submatrix

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix}_{2k \times 2k} \tag{11}$$

of *A* is unimodular.

Proof. See Theorem 2 in Rao (1997).

Theorem 3.2. If $A \in Z^{n \times n}$ is totally unimodular of full rank, then the integer *WZ* and *ZW* factorizations can always be obtained by pivoting. That is, there exists a row permutation matrix Π and the factors *W* and *Z* such that

$$\Pi \; A = WZ. \tag{12}$$

Proof. See Theorem 3 in Rao (1997).

Corollary 3.1. Every totally unimodular symmetric positive definite matrix has the integer *WZ* and *ZW* factorizations.

Now, we present a new interpretation of Theorem 3.1 based on the block *IABS* algorithm with blocksize equal to two. Then, we show how to compute the integer *WZ* factorization using *IABS* algorithm.

Theorem 3.3. Let $A \in Z^{n \times n}$ be unimodular. If Δ_k, $k=1,\ldots,\frac{n}{2}$ be unimodular then the block scaled *IABS* algorithm with parameter choices $H_1 = I$, $V_i = [v_{2i-1}, v_{2i}] = [e_i, e_{n-i+1}]$ and $Q_i = [q_{2i-1}, q_{2i}] = [e_i, e_{n-i+1}]$ is well defined and the implicit factorization $V^T AP$ with $p_i = H_i^T e_i, p_{n-i+1} = H_i^T e_{n-i+1}$, $i = 1,\ldots,\frac{n}{2}$ and $V = [V_1,\ldots,V_{\frac{n}{2}}]$ leads to an integer *WZ* factorization.

Proof. Let $H_1 = I$ and H_{i+1} defined by (7). Then, according to property **P3**, we have

$$H_{i+1} = \begin{bmatrix} 0 & 0 & 0 \\ K_i & I_{n-2i} & L_i \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

with $K_i, L_i \in Z^{n-2i,i}$. Let $\bar{P}_i = [\bar{p}_{2i-1}, \bar{p}_{2i}] = H_i^T [e_i, e_{n-i+1}]$, $\bar{P} = [\bar{P}_i, \ldots \bar{P}_{\frac{n}{2}}]$ and $V = [V_1,\ldots,V_{\frac{n}{2}}]$. Then, the integer block ABS algorithm produce $V^T A\bar{P} = L$, where L is a block lower triangular matrix. Now, we have

$$V^T A\bar{P} = L \Rightarrow A\bar{P}V^T = V^{-T} L V^T \Rightarrow AP = V^{-T} L V^T \Rightarrow A = WZ, \tag{14}$$

Where $P = (\bar{P}V^T)$ is an integer Z-matrix with 1's on diagonal and 0's on off diagonal and $W = V^{-T} L V^T$ is an integer W-matrix.

We observe that the first i rows and the last i rows of H_{i+1} are equal zero and we delete the rows. In doing this we use of the matrix E_i obtaining from I_n by deleting $i$ts the first i rows and the last i rows.

Here, we present an algorithm for computing the integer *WZ* factorization.

Algorithm 3.The integer *WZ* factorization.

Step 1: Let $H_1 = I$ and $i = 1$.
Step 2: Let $A_i = [a_i, a_{n-i+1}]$, $s_i = H_i A_i$ and

$$P_i = [p_i, p_{n-i+1}] = H_i^T [e_i, e_{n-i+1}].$$

Step 3: Let $Q_i = [e_i, e_{n-i+1}]$ and $F_i = Q_i^T S_i$. Construct E_i from I_i by deleting its the first *i* rows and the last *i* rows. Update H_i by

$$H_{i+1} = E_i (H_i - S_i (F_i^{-1}) P_i^T).$$

Step 4: Let $i = i + 1$. If $i \le \frac{n}{2}$ go to Step (2).

Step 5: Compute *AP* = *W*, where $P = [p_1, ..., p_n]$. Stop.

Theorem 3.4. Let *A* be totally tunimodular symmetric positive definite. Then, there exists a $Z^T XZ$ factorization for *A*, obtained by the *ABS* algorithm.

Proof. Consider the assumptions of Theorem 3.3 and let $V_i = P_i$, for $i = i = 1, \dots, s$. Then,

$$V^T AP = L \Rightarrow A = V^{-T} LP^{-1} = Z^T XZ \tag{15}$$

where *X* is an *X*-matrix.

4. *ZW* FACTORIZATION USING THE BLOCK SCALED *ABS* ALGORITHM

Now, the integer *ZW* factorization is presented as an alternative to the integer *WZ* factorization.

Definition 4.1. We say that a matrix *A* is factorized in the form integer *ZW* if

$$A = ZW, \tag{16}$$

where the matrices W is an integer W-matrix and Z is an integer Z-matrix.

Theorem 4.1. Let $A \in Z^{n \times n}$ be unimodular. The matrix A has an integer ZW factorization if and only if for every k, $k = 1,\ldots,s$, with $s = \left\lfloor \frac{n}{2} \right\rfloor$, if n is even, and $s = \left\lceil \frac{n}{2} \right\rceil$, if n is odd, the submatrix

$$\Lambda_k = \begin{pmatrix} a_{s-k+1,s-k+1} & \cdots & a_{s-k+1,s+k} \\ \vdots & \cdots & \vdots \\ a_{s+k,s-k+1} & \cdots & a_{s+k,s+k} \end{pmatrix} \tag{17}$$

of A is unimodular.

Proof. See Theorem 2 in Rao (1997) replacing Δ_i by Λ_i.

Here, we compute the integer ZW factorization using the block integer *ABS* algorithm.

Theorem 4.2. Let $A \in Z^{n \times n}$ be unimodular. If Λ_k, $k = 1,\ldots,\frac{n}{2}$ be unimodular then the block *IABS* algorithm with parameter choices $H_1 = I$, $V_i = [v_{2i-1}, v_{2i}] = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$ and $Q_i = [q_{2i-1}, q_{2i}] = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$ is well defined and the implicit factorization $V^T AP$ with $p_{\frac{n}{2}-i+1} = H_i^T e_{\frac{n}{2}-i+1}$, $p_{\frac{n}{2}+i} = H_i^T e_{\frac{n}{2}+i}$, $i = 1,\ldots,\frac{n}{2}$ and $V = [V_1,\ldots,V_{\frac{n}{2}}]$ leads to an integer ZW factorization.

Proof. Let $H_1 = I$ and H_{i+1} defined by (7). Then, according to property **P3**, we have

$$H_{i+1} = \begin{bmatrix} I_i & K_{2i} & 0 \\ 0 & 0 & 0 \\ 0 & L_{2i} & I_i \end{bmatrix} \tag{18}$$

with $K_i, L_i \in Z^{n-2i,i}$. Let $\bar{P}_i = [\bar{p}_{2i-1}, \bar{p}_{2i}] = H_i^T [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$, $\bar{P} = [\bar{P}_i, \ldots, \bar{P}_{\frac{n}{2}}]$ and $V = [V_1, \ldots, V_{\frac{n}{2}}]$. Then, the integer block ABS algorithm produce $V^T A\bar{P} = L$, where L is a lower triangular matrix. Now, we have

$$V^T A\bar{P} = L \Rightarrow A\bar{P}V^T = V^{-T}LV^T \Rightarrow AP = V^{-T}LV^T \Rightarrow A = ZW \tag{19}$$

where, $P = (\bar{P}V^T)$ is an integer W-matrix with 1's on diagonal and 0's on off diagonal and $Z = V^{-T}LV^T$ is an integer Z-matrix.

We observe that the first $(\frac{n}{2} - i + 1)$ th to $(\frac{n}{2} + i)$ th rows of H_{i+1} are equal zero and we delete the rows. In doing this we use of the matrix E_i obtaining from I_n by deleting $(\frac{n}{2} - i + 1)th$ until $(\frac{n}{2} + i)th$ rows.

Here, we present an algorithm for computing the integer ZW factorization.

Algorithm 4. The integer *ZW* factorization.

Step 1: Let $H_1 = I$ and $i = 1$.

Step 2: Let $A_i = [a_{\frac{n}{2}-i+1}, a_{\frac{n}{2}+i}]$, $S_i = H_i A_i$ and

$$P_i = [p_{\frac{n}{2}-i+1}, p_{\frac{n}{2}+i}] = H_i^T [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$$

Step 3: Let $Q_i = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$ and $F_i = Q_i^T S_i$. Construct E_i from I_i by deleting $(\frac{n}{2} - i + 1)th$ until $(\frac{n}{2} + i)th$ rows. Update H_i by

$$H_{i+1} = E_i(H_i - S_i(F_i^{-1})P_i^T).$$

Step 4: Let $i = i + 1$. If $i \leq \frac{n}{2}$ go to Step (2).

Step 5: Compute $AP = Z$, where $P=[p_1,...,p_n]$. Stop.

Theorem 4.3. Let A be totally unimodular symmetric positive definite. Then, there exists a W^TXW factorization for A, obtained by the *ABS* algorithm.

Proof. Consider the assumptions of Theorem 4.2 and let $V_i = P_i$, for $i=1,...,s.$ Then,

$$V^T AP = L \Rightarrow A = V^{-T} LP^{-1} = W^T XW \qquad (20)$$

where X is an X-matrix.

For computing the integer *WZ* (*ZW*) factorization by the Algorithm 3 (4), in the kth step we need to store the $(2i-1)\times 2$ nonzero elements of submatrix of P_i, the $(n-2i+2)\times 2$ nonzero elements of submatrix of S_i^T and 4 for F, used to update H_i. Thus the storage required is the storage of A, $2n$ for S_i, 4 for F plus $\sum_{i=1}^{n/2} 2(2i-1) = n^2/2$ for the matrix P.

We observe that no computations are required for evaluating P_i. In the evaluation of H_{i+1} no more than $2(n-2i+2)(2i-2)$ multiplications are required for computing $H_i A_i$, since unit submatrix I_{n-2i-2} in H_i, 2 multiplications and 4 divisions are required for computing F^{-1}, no more than $(2i-1)$ multiplications and $(2i-1)$ divisions are required for computing $F^{-1}P_i$, no more than $2(n-2i+2)(2i-1)$ multiplications are required for computing the nonzero elements of $S_i F^{-1} P_i^T$. Then the computing cost follows by summing all terms with no more than $\frac{n^3}{3} + O(n^2)$.

5. A NUMERICAL ILLUSTRATION

Here, we present a numerical illustration of the Algorithm 3 for computing an integer *WZ* factorization.

Example: Considering the following matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

Upon an application of Algorithms 3 for computing the integer *WZ* factorization we have,

$$P = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & -1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -8 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -8 & -1 & -3 & 1 & 1 \end{bmatrix}.$$

which is a *Z*-matrix and

$$W = AP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & -3 & 0 & 0 & 4 & 0 & -1 \\ -1 & -1 & -4 & -21 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 & 0 & 6 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

which is a *W*-matrix. Therefore,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & -3 & 0 & 0 & 4 & 0 & -1 \\ -1 & -1 & -4 & -21 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 & 0 & 6 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & -2 & -1 & 0 \\ 0 & 1 & -3 & 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}$$

6. CONCLUSION

We provided the conditions for the existence of the integer *WZ* and the integer *ZW* factorizations of a unimodular integer matrix. Then, we presented efficient algorithms in computation and storage for computing the integer *WZ* and *ZW* factorizations of an integer matrix and the integer $Z^T XZ$ and $W^T XW$ factorizations of a totally unimodular symmetric positives definite matrix using the integer *ABS* algorithm.

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REFERENCES

Abaffy, J. and Broyden, C. G. (1984).A class of direct methods for linear equations.*Numer.Math*.**45**: 361–378.

Abaffy,J. and Galantai, A. (1986). Conjugate direction methods for linear and nonlinear systems of algebraic equations.*Colloq. Math. Soc. Janos Bolyai.Numerical Methods. Miskolc*.**50**: 481-502.

Abaffy, J. and Spedicato, E. (1989). *ABS Projection Algorithms, Mathematical Techniques for Linear and Nonlinear Equations*.Chichester:Ellis Horwood.

Bodon, E. (2001).On the block implicit LU algorithmforlinear systems of equations.*Mathematical Notes. Miskolc*.**2**(1): 11-29.

Esmaeili.H., Mahdavi-Amiri, N. and Spedicato,E. (2001). Generationg the integer null space and conditions for determination of an integer basis using the ABS algorithms.*Bulletin of the Iranian Mathematical Society*.**27**:1–18.

Esmaeili, H., Mahdavi-Amiri, N. and Spedicato, E. (2001). A class of ABS algorithms for Diophantine linear systems.*Numer.Math.***90**:101–115.

Evans, D. J. and Hatzopoulos, M. (1979). A parallel linear system solver*.,* *International Journal of Computer Mathematics*.7: 227–238.

Evans, D. J. (1993a).Implicit matrix elimination schemes. *Int. J. Computer Math.***48**:229-237.

Evans, D. J. and Abdullah,R. (1994).The parallel implicit elimination (PIE) method for the solution of linear systems.*Parallel Algorithms and Applications*.**4**: 153-162.

Evans, D. J. (1993b). Parallel implicit schemes for the solution of linear systems. *in: G. Winter Althaus and E. Spedicato (eds.). Algorithms for large Scale Linear Algebraic Systems. NATO ASI Series, Series C. Mathematical and Physical Sciences. Kluwer*. **508**: 37-54.

Evans, D. J. (1999).The Cholesky QIF algorithm for solving symmetric linear systems. *International Journal of Computer Mathematics*.**72**: 283–285.

Galantai, A. (2001). Rank reduction, factorization and conjugation. *Linear and Multilinear Algebra*.**49**: 195-207.

Galantai, A. (2003). Projection methods for linear and nonlinear equations. *Dissertation submitted to the Hungarian Academy of Sciences for the degree"MTADoktora". University of Miskolc*.

Galantai, A. (2004). *Projectors and Projection Methods*.Kluwer.

Golpar-Raboky, E. and Mahdavi-Amiri, N. (2012).Diophantine quadratic equation and Smith normal form using scaled extended integer ABS algorithms. *Journal of Optimization Theory and Applications*. **152**(1): 75–96 .

Khorramizadeh, M. and Mahdavi-Amiri, N. (2009). Integer extended ABS algorithms and possible control of intermediate results for linear Diophantine systems.*4OR*.**7**: 145–167.

Khorramizadeh, K. and Mahdavi-Amiri, N. (2008).On solving linear Diophantine systems using generalized Rosser's algorithm. *Bulletin of the Iranian Mathematical Society*. **34**(2): 1–25.

Pohst, M. (1993).Computational Algebraic Number Theory. Boston: *BirkhauserVerlag*.

Rao, S.C.S. (1997). Existence and uniqueness of WZ factorization. *Parallel Comput*. **23**:1129–1139.

Spedicato, E., Bodon, E., Del Popolo, A. and Mahdavi-Amiri, N. (2003). ABS methods and ABSPACK for linear systems and optimization.*A review.4OR*. **1**: 51-66.

Spedicato, E, Bodon, E., Zunquan, X. and Mahdavi-Amiri, N. (2010). ABS methods for continous and integer linear equations and optimization. *CEJOR*. **18**: 73-95.

Spedicato, E., Xia, Z. and. Zhang, L. (1997). The implicit LX method of the ABS class.*Optimization Methods and Software*. **8**: 99–110.

Wedderburn, J. H. M. (1934). *Lectures on Matrices*. Colloquium Publications. New York: American Mathematical Society.

Zou, M. and Xia, Z. (2005).ABS algorithms for Diophantine linear equations and linear LP problems. *J. Appl. Math. and Computing*. **17**(1-2):93–107.