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Ranking Fuzzy Numbers based on Sokal and Sneath Index with Hurwicz Criterion

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ABSTRACT

Ranking of fuzzy numbers is an important procedure for many applications in fuzzy theory, in particular, decision-making. In this paper, we propose a novel method for ranking fuzzy numbers using Sokal and Sneath set theoretic index. The fuzzy maximum, fuzzy minimum, fuzzy evidences and fuzzy total evidences are obtained in determining the ranking. The Hurwicz criterion which considers all types of decision makers' perspective is employed in aggregating the fuzzy total evidences. The rationality properties of the proposed method are presented. Moreover, five numerical examples are presented to illustrate the advantages of the proposed method. The ranking results show that the proposed method can overcome certain shortcomings that exist in the previous ranking methods.

Keywords: Decision-making, Hurwicz criterion, ranking fuzzy numbers, Sokal and Sneath index

1. INTRODUCTION

Ranking of fuzzy numbers is an important procedure for many applications in fuzzy theory such as in approximate reasoning, decision-making, optimization and other usages. In fuzzy decision analysis, fuzzy numbers are employed to describe the performance of alternatives and the selection of alternatives will eventually lead to the ranking of corresponding fuzzy numbers. However, ranking of fuzzy numbers is not an easy task since fuzzy numbers are represented by possibility distributions and they can overlap with each other.

Since Jain (1976) first presented the concept of ranking fuzzy numbers, various methods of ranking fuzzy numbers have been developed but no method can rank fuzzy numbers satisfactorily in all cases and situations. Some methods produce non-discriminate and non-intuitive results, limited to normal and triangular types of fuzzy numbers and only consider neutral decision makers' perspective. There are also methods that produce different ranking results for the same situations and some have the difficulty of interpretation.

An early review on ranking fuzzy numbers has been done by Bortolan and Degani (1985), followed by Chen and Hwang (1992) and Wang and Kerre (1996). In 1998, Cheng proposed a distance index based on the centroid concept and CV index. The distance index has improved Yagers' index (1980), while the CV index has improved Lee and Li's (1988) approach. However, in some situations, the ranking result by the distance index contradicts with the result by the CV index.

Thus, to overcome the problems, Chu and Tsao (2002) proposed an area between the centroid point and original point as the ranking index. Chen and Chen (2007) then, found that Cheng's (1998) distance index, Chu and Tsao's (2002) and Yagers' (1980) methods cannot rank correctly two fuzzy numbers having the same mode and symmetric spread. Thus, Chen and Chen (2007) proposed a new ranking approach using the score index concept. However, Chen and Chen's (2007) method is only limited to trapezoidal type of fuzzy numbers and does not cater for general fuzzy numbers.

In other studies by Yao and Wu (2000) and Abbasbandy and Asady (2006), they proposed signed distance method for ranking fuzzy numbers. Furthermore, Asady and Zendehnam (2007) proposed distance minimization method for ranking fuzzy numbers. However, Yao and Wu's (2000), Abbasbandy and Asady's (2006) and Asady and Zendehnam's (2007) methods are only limited to normal fuzzy numbers and are found to produce non-discriminative ranking result for fuzzy numbers having the same mode and symmetric spread. In a different study by Setnes and Cross (1997), they proposed Jaccard index with mean aggregation concept for ranking fuzzy numbers. However, their methods are only applicable for normal fuzzy numbers, only consider neutral decision makers' perspective and also cannot distinguish the ranking of fuzzy numbers having the same mode and symmetric spread.

In this paper, a new method for ranking fuzzy numbers based on Sokal and Sneath index and Hurwicz criterion is proposed. The Sokal and Sneath is a set theoretic type of similarity measure index which is commonly used in pattern recognition and classification for population diversity, and Hurwicz is a criterion for decision-making that compromises between the optimistic and pessimistic criteria. Thus, the proposed ranking method considers all types of decision makers' perspective such as optimistic, neutral and pessimistic, which is crucial in solving decision-making problems. The proposed method can overcome certain shortcomings that exist in the previous ranking methods.

The paper is organized as follows. Section 2 briefly reviews the preliminary concepts and definitions. In Section 3, we propose the Sokal and Sneath index with Hurwicz criterion for ranking fuzzy numbers. The rationality properties of the proposed ranking method are presented in Section 4. Section 5, presents five numerical examples to illustrate the advantages of the proposed method. Lastly, the paper is concluded in Section 6.

2. PRELIMINARIES

In this section, some basic concepts and definitions on fuzzy numbers are reviewed from the literature.

Definition 1

A fuzzy number is a fuzzy set in the universe of discourse X with the membership function defined as (Dubois and Prade (1980));

$$\mu_A(x) = \begin{cases} \mu_A^L(x) & , a \le x \le b \\ w & , b \le x \le c \\ \mu_A^R(x) & , c \le x \le d \\ 0 & , \text{otherwise} \end{cases}$$

where $\mu_A^L : [a,b] \to [0,w]$, $\mu_A^R : [c,d] \to [0,w]$, $w \in (0,1]$, μ_A^L and μ_A^R denote the left and the right membership functions of the fuzzy number A . The membership function μ_A of a fuzzy number A has the following properties:

(1) μ_A is a continuous mapping from the universe of discourse X to $[0,w]$.
(2) $\mu_A(x) = 0$ for $x < a$ and $x > d$.
(3) $\mu_A(x)$ is monotonic increasing in $[a,b]$.
(4) $\mu_A(x) = w$ for $[b,c]$.
(5) $\mu_A(x)$ is monotonic decreasing in $[c,d]$.

If the membership function $\mu_A(x)$ is a piecewise linear, then A is called as a trapezoidal fuzzy number with membership function defined as

$$\mu_{\tilde{A}_1}(x) = \begin{cases} w\left(\dfrac{x-a}{b-a}\right) & , a \le x \le b \\ w & , b \le x \le c \\ w\left(\dfrac{d-x}{d-c}\right) & , c \le x \le d \\ 0 & , \text{otherwise} \end{cases}$$

and denoted as $A = (a,b,c,d;w)$. If $b = c$, then the trapezoidal becomes a triangular fuzzy number denoted as $A = (a,b,d;w)$.

Definition 2

Let A_1 and A_2 be two fuzzy numbers with $A_{1_\alpha} = [a_\alpha^-, a_\alpha^+]$ and $A_{2\alpha} = [b_\alpha^-, b_\alpha^+]$ be their α-cuts with $\alpha \in [0,1]$. The fuzzy maximum of A_1 and A_2 by the α-cuts method is defined as (Kaufmann and Gupta (1985));

$$\left[MAX\left(A_1, A_2\right)\right]_\alpha = \left[\max\left(a_\alpha^-, b_\alpha^-\right), \max\left(a_\alpha^+, b_\alpha^+\right)\right].$$

The fuzzy minimum of A_1 and A_2 is defined as

$$\left[MIN\left(A_1, A_2\right)\right]_\alpha = \left[\min\left(a_\alpha^-, b_\alpha^-\right), \min\left(a_\alpha^+, b_\alpha^+\right)\right].$$

Definition 3

Let $A_1 = (a_1, b_1, c_1, d_1; h_1)$ and $A_2 = (a_2, b_2, c_2, d_2; h_2)$ be two trapezoidal fuzzy numbers. The fuzzy maximum of A_1 and A_2 by the second function principle is defined as (Chen and Hsieh (1998));

$$MAX\left(A_1, A_2\right) = \left(a, b, c, d; h\right)$$

where

$$h = \min\{h_1, h_2\},$$
$$T = \{\max(a_1, a_2), \max(a_1, d_2), \max(d_1, a_2), \max(d_1, d_2)\},$$
$$T_1 = \{\max(b_1, b_2), \max(b_1, c_2), \max(c_1, b_2), \max(c_1, c_2)\},$$
$$a = \min T,\ b = \min T_1,\ c = \min T_1,\ d = \max T,\ \min T \le \min T_1 \text{ and}$$
$$\max T_1 \le \max T.$$

The fuzzy minimum of A_1 and A_2 is defined as,

$$MIN\left(A_1, A_2\right) = \left(a, b, c, d; h\right)$$

where

$$h = \min\{h_1, h_2\},\ \ T = \{\min(a_1, a_2), \min(a_1, d_2), \min(d_1, a_2), \min(d_1, d_2)\},$$

$$T_1 = \{\min(b_1, b_2), \min(b_1, c_2), \min(c_1, b_2), \min(c_1, c_2)\},\ a = \min T,$$
$$b = \min T_1,\ c = \max T_1,\ d = \max T,\ \min T \le \min T_1 \text{ and } \max T_1 \le \max T.$$

Definition 4

The scalar cardinality of a fuzzy number A in the universe of discourse X is defined as (Zwick *et al.* (1987)),

$$|A| = \int_X \mu_A(x)dx.$$

Definition 5

For fuzzy numbers A_i, A_j and A_k, a fuzzy preference P is called w3-transitive if and only if (Wang and Ruan (1995));

$$P(A_i, A_j) > P(A_j, A_i) \text{ and } P(A_j, A_k) > P(A_k, A_j) \text{ implies}$$
$$P(A_i, A_k) > P(A_k, A_i).$$

3. SOKAL AND SNEATH RANKING INDEX WITH HURWICZ CRITERION

Based on the psychological ratio model of similarity from Tversky (1977) which is defined as

$$S_{\alpha,\beta}(X,Y) = \frac{f(X \cap Y)}{f(X \cap Y) + \alpha.f(X - Y) + \beta.f(Y - X)},$$

various index of similarity measures have been proposed. For $\alpha = 2$ and $\beta = 2$, the ratio model of similarity becomes the Sokal and Sneath index which is defined as $S_{2,2}(X,Y) = \frac{f(X \cap Y)}{2.f(X \cup Y) - f(X \cap Y)}$. Typically, the function f is taken to be the cardinality function. The objects X and Y described by the features are replaced with fuzzy numbers A and B which are described by the membership functions. The fuzzy Sokal and Sneath index is defined as $S_{SS}(A,B) = \frac{|A \cap B|}{2.|A \cup B| - |A \cap B|}$ where $|A|$ denotes the scalar cardinality of A, $\cap$ and $\cup$ are the t-norm and s-norm respectively. We propose fuzzy Sokal and Sneath ranking index with Hurwicz criterion as follows:

Step 1:

For each pair of the fuzzy numbers A_i and A_j, find the fuzzy maximum and fuzzy minimum of A_i and A_j. The fuzzy maximum and fuzzy minimum can be obtained by the α-cuts method for normal fuzzy numbers and, the second function principle for non-normal fuzzy numbers.

Step 2:

Calculate the evidences of $E(A_i \succ A_j)$, $E(A_j \prec A_i)$, $E(A_j \succ A_i)$ and $E(A_i \prec A_j)$ which are defined based on fuzzy Sokal and Sneath index as, $E(A_i \succ A_j) = S_{SS}(MAX(A_i, A_j), A_i)$, $E(A_j \prec A_i) = S_{SS}(MIN(A_i, A_j), A_j)$, $E(A_j \succ A_i) = S_{SS}(MAX(A_i, A_j), A_j)$ and $E(A_i \prec A_j) = S_{SS}(MIN(A_i, A_j), A_i)$ where $S_{SS}(A_i, A_j) = \dfrac{|A_i \cap A_j|}{2|A_i \cup A_j| - |A_i \cap A_j|}$ is the fuzzy Sokal and Sneath index and $|A_i|$ denotes the scalar cardinality of fuzzy number A_i. To simplify, C_{ij} and c_{ji} are used to represent $E(A_i \succ A_j)$ and $E(A_j \prec A_i)$, respectively. Likewise, C_{ji} and c_{ij} are used to denote $E(A_j \succ A_i)$ and $E(A_i \prec A_j)$ respectively.

Step 3:

Calculate the total evidences $E_{total}(A_i \succ A_j)$ and $E_{total}(A_j \succ A_i)$ which are defined based on the Hurwicz criterion concept as $E_{total}(A_i \succ A_j) = \beta C_{ij} + (1-\beta)c_{ji}$ and $E_{total}(A_j \succ A_i) = \beta C_{ji} + (1-\beta)c_{ij}$. $\beta \in [0, 0.5)$, $\beta = 0.5$ and $\beta \in (0.5, 1]$ represent pessimistic, neutral and optimistic criteria respectively. To simplify, $E_{SS}(A_i, A_j)$ and $E_{SS}(A_j, A_i)$ are used to represent $E_{total}(A_i \succ A_j)$ and $E_{total}(A_j \succ A_i)$, respectively.

Step 4:

For each pair of the fuzzy numbers, compare the total evidences in Step 3 which will result the ranking of the two fuzzy numbers A_i and A_j as follows:

(i) $A_i \succ A_j$ if and only if $E_{SS}\left(A_i, A_j\right) > E_{SS}\left(A_j, A_i\right)$.

(ii) $A_i \prec A_j$ if and only if $E_{SS}\left(A_i, A_j\right) < E_{SS}\left(A_j, A_i\right)$.

(iii) $A_i \approx A_j$ if and only if $E_{SS}\left(A_i, A_j\right) = E_{SS}\left(A_j, A_i\right)$.

Step 5:

Check the transitivity of $E_{SS}\left(A_i, A_j\right)$ by using the w3-transitivity from Wang and Ruan (1995).

Step 6:

For *n* fuzzy numbers with transitive pair wise ranking, do the total ordering. While for non-transitive pair wise ranking, use the size of dominated class method from Cross and Setnes (1998).

4. RATIONALITY PROPERTIES

We consider the rationality properties for the ordering approaches by Wang and Kerre (2001). The properties are presented in Table 1 with *M* be the ordering index, S is the set of fuzzy quantities for which index *M* can be applied, X is a finite subset of S and $A_1, A_2, A_3 \in X$.

TABLE 1: Rationality Properties for Ordering Indices (Wang and Kerre (2001))

Axioms	Properties
W_1	For an arbitrary finite subset *X* of *S* and $A_1 \in X$, $A_1 \gtrsim A_1$ by *M* on *X*.
W_2	For an arbitrary finite subset *X* of *S* and $\left(A_1, A_2\right) \in X^2$, $A_1 \gtrsim A_2$ and $A_2 \gtrsim A_1$ by *M* on *X*, we should have $A_1 \approx A_2$ by *M* on *X*.
W_3	For an arbitrary finite subset *X* of *S* and $\left(A_1, A_2, A_3\right) \in X^3$, $A_1 \gtrsim A_2$ and $A_2 \gtrsim A_3$ by *M* on *X*, we should have $A_1 \gtrsim A_3$ by *M* on *X*.
W_4	For an arbitrary finite subset *X* of *S* and $\left(A_1, A_2\right) \in X^2$, $\inf \operatorname{supp}\left(A_1\right) > \sup \operatorname{supp}\left(A_2\right)$, we should have $A_1 \gtrsim A_2$ by *M* on *X*.

TABLE 1 (continued): Rationality Properties for Ordering Indices (Wang and Kerre (2001))

Axioms	Properties
$W_4^{'}$	For an arbitrary finite subset X of S and $(A_1, A_2) \in X^2$, $\inf \operatorname{supp}(A_1) > \sup \operatorname{supp}(A_2)$, we should have $A_1 \succ A_2$ by M on X.
W_5	Let S and S' be two arbitrary finite sets of fuzzy quantities in which M can be applied, A_1 and A_2 are in $S \cap S'$. We obtain the ranking order $A_1 \succ A_2$ by M on S' if and only if $A_1 \succ A_2$ by M on S.
W_6	Let A_1, A_2, $A_1 + A_3$ and $A_2 + A_3$ be elements of S. If $A_1 \gtrsim A_2$ by M on $\{A_1, A_2\}$, then $A_1 + A_3 \gtrsim A_2 + A_3$ by M on $\{A_1 + A_3, A_2 + A_3\}$.
$W_6^{'}$	Let A_1, A_2, $A_1 + A_3$ and $A_2 + A_3$ be elements of S. If $A_1 \succ A_2$ by M on $\{A_1, A_2\}$, then $A_1 + A_3 \succ A_2 + A_3$ by M on $\{A_1 + A_3, A_2 + A_3\}$ for $A_3 \neq 0$.
W_7	Let A_1, A_2, A_1A_3 and A_2A_3 be elements of S and $A_3 \geq 0$. If $A_1 \gtrsim A_2$ by M on $\{A_1, A_2\}$, then $A_1A_3 \gtrsim A_2A_3$ by M on $\{A_1A_3, A_2A_3\}$.

Note: $W_4^{'}$ is stronger than W_4 which means that the ranking index meets W_4 if it meets $W_4^{'}$ (Wang and Kerre (2001)).

Theorem

The function $E_{SS}(A_i, A_j)$ has the properties of W_1, W_2, W_4, $W_4^{'}$ and W_5.

Proof.

Let A_i and A_j be two fuzzy numbers with $E_{SS}(A_i, A_j)$ be the total evidences $E_{total}(A_i \succ A_j)$ of Sokal and Sneath index. Then,

$$E_{SS}(A_i, A_i) = \beta \frac{|MAX(A_i, A_i) \cap A_i|}{2|MAX(A_i, A_i) \cup A_i| - |MAX(A_i, A_i) \cap A_i|} + (1-\beta) \frac{|MIN(A_i, A_i) \cap A_i|}{2|MIN(A_i, A_i) \cup A_i| - |MIN(A_i, A_i) \cap A_i|}$$

$$= \beta \frac{|A_i|}{|A_i|} + (1-\beta) \frac{|A_i|}{|A_i|} = \beta(1) + (1-\beta)(1) = 1$$

Obviously, $E_{SS}(A_i, A_i) \geq E_{SS}(A_i, A_i)$ and, thus, $A_i \gtrsim A_i$ by E_{SS}.
Hence, E_{SS} satisfies axiom W_1.

Now,

$A_i \gtrsim A_j$ implies $E_{SS}\left(A_i, A_j\right) \geq E_{SS}\left(A_j, A_i\right)$ and
$A_j \gtrsim A_i$ implies $E_{SS}\left(A_j, A_i\right) \geq E_{SS}\left(A_i, A_j\right)$.

By anti-symmetric rules, clearly $E_{SS}\left(A_i, A_j\right) = E_{SS}\left(A_j, A_i\right)$ which implies $A_i \approx A_j$ by E_{SS} and axiom W_2 is satisfied.

Two cases are considered for showing that axioms W_4 and W_4' are satisfied. Firstly, assume X is the universe of discourse with $\text{hgt}\left(A_i\right) \geq \text{hgt}\left(A_j\right)$ where $\text{hgt}\left(A_i\right)$ denotes the height of fuzzy number A_i.

If $\inf \text{supp}\left(A_i\right) > \sup \text{supp}\left(A_j\right)$, clearly, we obtain the following:

$MAX\left(A_i, A_j\right) \subseteq A_i$ since $\mu_{MAX}(x) \leq \mu_{A_i}(x)$ for $\forall x \in X$
(Dubois and Prade, (1980)).

$MIN\left(A_i, A_j\right) = A_j$, $MAX\left(A_i, A_j\right) \cap A_j = \phi$, $MIN\left(A_i, A_j\right) \cap A_i = \phi$.

Thus,

$$E_{SS}\left(A_i, A_j\right) = \beta \frac{\left|MAX\left(A_i, A_j\right) \cap A_i\right|}{2\left|MAX\left(A_i, A_j\right) \cup A_i\right| - \left|MAX\left(A_i, A_j\right) \cap A_i\right|} + (1-\beta) \frac{\left|MIN\left(A_i, A_j\right) \cap A_j\right|}{2\left|MIN\left(A_i, A_j\right) \cup A_j\right| - \left|MIN\left(A_i, A_j\right) \cap A_j\right|}$$

$$= \beta \frac{\left|MAX\left(A_i, A_j\right)\right|}{2\left|A_i\right| - \left|MAX\left(A_i, A_j\right)\right|} + (1-\beta) \frac{\left|A_j\right|}{2\left|A_j\right| - \left|A_j\right|}$$

$$\leq \beta \frac{\left|MAX\left(A_i, A_j\right)\right|}{2\left|MAX\left(A_i, A_j\right)\right| - \left|MAX\left(A_i, A_j\right)\right|} + (1-\beta)(1) = \beta + (1-\beta) = 1 \ .$$

Obviously, $E_{SS}\left(A_i, A_j\right) > 0$ and thus, $0 < E_{SS}\left(A_i, A_j\right) \leq 1$.

$$E_{SS}\left(A_j, A_i\right) = \beta \frac{\left|MAX\left(A_i, A_j\right) \cap A_j\right|}{2\left|MAX\left(A_i, A_j\right) \cup A_j\right| - \left|MAX\left(A_i, A_j\right) \cap A_j\right|} + (1-\beta) \frac{\left|MIN\left(A_i, A_j\right) \cap A_i\right|}{2\left|MIN\left(A_i, A_j\right) \cup A_i\right| - \left|MIN\left(A_i, A_j\right) \cap A_i\right|}$$

$$= \beta(0) + (1-\beta)(0) = 0.$$

Thus, $E_{SS}\left(A_i, A_j\right) > E_{SS}\left(A_j, A_i\right)$ which implies $A_i \succ A_j$ by E_{SS} and axiom W_4' is satisfied.

Hence, axiom W_4 is also satisfied.

Next, assume $\text{hgt}\left(A_i\right) < \text{hgt}\left(A_j\right)$.

If $\inf \text{supp}\left(A_i\right) > \sup \text{supp}\left(A_j\right)$, clearly, we obtain the following:

$$MAX\left(A_i, A_j\right) = A_i,$$
$$MIN\left(A_i, A_j\right) \subseteq A_j \quad \text{since} \quad \mu_{MIN}(x) \leq \mu_{A_j}(x) \text{ for } \forall x \in X.$$
$$MAX\left(A_i, A_j\right) \cap A_j = \phi,\ MIN\left(A_i, A_j\right) \cap A_i = \phi.$$

Thus,

$$E_{SS}\left(A_i, A_j\right) \leq \beta(1) + (1-\beta) \frac{\left|MIN\left(A_i, A_j\right)\right|}{2\left|MIN\left(A_i, A_j\right)\right| - \left|MIN\left(A_i, A_j\right)\right|} = 1,$$ and the result follows as in the case of $\text{hgt}\left(A_i\right) \geq \text{hgt}\left(A_j\right)$.

Assume that A_i and A_j are two fuzzy numbers in $S \cap S'$, where S and S' are two arbitrary finite sets of fuzzy numbers.

The ranking of A_i and A_j is solely influenced by $E_{SS}\left(A_i, A_j\right)$ and $E_{SS}\left(A_j, A_i\right)$ where

$$E_{SS}(A_i,A_j)=\beta\frac{|MAX(A_i,A_j)\cap A_i|}{2|MAX(A_i,A_j)\cup A_i|-|MAX(A_i,A_j)\cap A_i|}+(1-\beta)\frac{|MIN(A_i,A_j)\cap A_j|}{2|MIN(A_i,A_j)\cup A_j|-|MIN(A_i,A_j)\cap A_j|}$$

and

$$E_{SS}(A_j,A_i)=\beta\frac{|MAX(A_i,A_j)\cap A_j|}{2|MAX(A_i,A_j)\cup A_j|-|MAX(A_i,A_j)\cap A_j|}+(1-\beta)\frac{|MIN(A_i,A_j)\cap A_i|}{2|MIN(A_i,A_j)\cup A_i|-|MIN(A_i,A_j)\cap A_i|}$$

The operations only involved fuzzy numbers A_i and A_j (not involve any other fuzzy numbers in S or S'), and this ensures the same ranking order if it is based on S and S' and, thus, axiom W_5 is satisfied.

5. NUMERICAL EXAMPLES

In this section, five sets of numerical examples are presented to illustrate the validity and advantages of the proposed method.

Example 1

Consider the data used in Sun and Wu (2006), i.e., two fuzzy numbers $A_1=\left(0,\frac{3}{8},3\right)$ and $A_2=\left(\frac{1}{4},\frac{3}{2},\frac{5}{3},9\right)$ as shown in Figure 1.

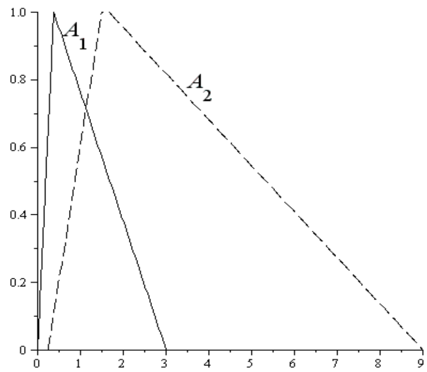

Figure 1: Fuzzy Numbers in Example 1

Intuitively, the ranking order is $A_1 \prec A_2$. However, by the fuzzy simulation analysis from Sun and Wu (2006), the ranking order is $A_2 \prec A_1$, which is unreasonable. By the proposed method, $E_{SS}(A_1, A_2) = 0.109$ and $E_{SS}(A_2, A_1) = 1$, therefore the ranking order is $A_1 \prec A_2$ regardless of the decision makers' perspective. This result is consistent with human intuition.

Example 2

Consider the data used in Wang *et al.* (2009), i.e., two triangular fuzzy numbers $A_1 = (3,6,9)$ and $A_2 = (5,6,7)$ as shown in Figure 2.

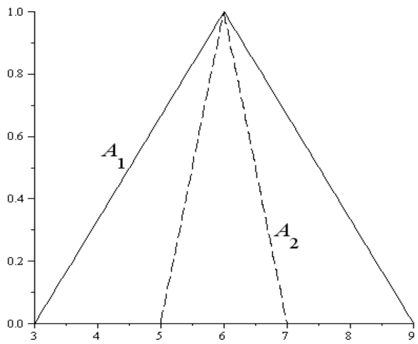

Figure 2: Fuzzy Numbers in Example 2

Since fuzzy numbers A_1 and A_2 have the same mode and symmetric spread, a number of the existing ranking methods cannot discriminate them, such as Chen (1985), Setnes and Cross (1997), Yao and Wu (2000), Chu and Tsao (2002), Abbasbandy and Asady (2006) with $p = 1$, Asady and Zendehnam (2007), and Wang and Lee (2008). The inconsistent results are also produced using distance index and CV index of Cheng's (1998) method. Moreover, Wang *et al.*'s (2005) method produces $A_1 \succ A_2$, while Wang *et al.*'s (2009) produces $A_2 \succ A_1$. By the proposed method, we obtain, $E_{SS}(A_1, A_2) = 0.167\beta + 0.333$ and $E_{SS}(A_2, A_1) = -0.167\beta + 0.5$.

Thus, the ranking order is
$$\begin{array}{ll} A_1 \prec A_2 & ,\beta \in [0, 0.5) \\ A_1 \approx A_2 & ,\beta = 0.5 \\ A_1 \succ A_2 & ,\beta \in (0.5, 1] \end{array},$$

where $A_1 \prec A_2$ for pessimistic decision makers, $A_1 \approx A_2$ for neutral decision makers, and $A_1 \succ A_2$ for optimistic decision makers. The ranking result is affected by decision makers' perspective and this shows that the proposed method has strong discrimination power. The result is also consistent with Wang and Luo's index (2009).

Example 3

Consider the data used in Wang and Lee (2008), i.e., two trapezoidal fuzzy numbers $A_1 = (6,7,9,10;0.6)$ and $A_2 = (5,7,9,10;1)$ as shown in Figure 3. Some of the existing ranking methods such Setnes and Cross (1997), Yao and Wu (2000), Wang *et al.* (2005), Abbasbandy and Asady (2006), Asady and Zendehnam (2007), Wang *et al.* (2009) and, Wang and Luo (2009) can only rank normal fuzzy numbers and, thus, fail to rank the fuzzy numbers A_1 and A_2. Moreover, Chu and Tsao (2002) rank them as $A_1 \prec A_2$, while Cheng's distance index and Wang and Lee's (2008) index rank them as $A_2 \prec A_1$.

By the proposed method, we have $E_{SS}(A_1, A_2) = 0.429 + 0.571\beta$ and $E_{SS}(A_2, A_1) = 0.75 - 0.404\beta$, thus, obtain the ranking result as $A_1 \prec A_2$ for $\beta \in [0,0.329)$, $A_1 \approx A_2$ for $\beta = 0.329$ and $A_1 \succ A_2$ for $\beta \in (0.329,1]$. Thus, the ranking result is affected by decision makers' perspective and this shows that the equal ranking result does not necessarily occur for neutral decision makers.

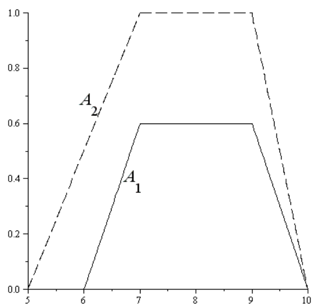

Figure 3: Fuzzy Numbers in Example 3

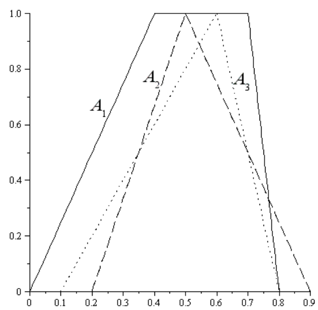

Figure 4: Fuzzy Numbers in Example 4

Example 4

Consider the data used in Abbasbandy and Hajjari (2009), i.e., a trapezoidal fuzzy number and two triangular fuzzy numbers, $A_1 = (0,0.4,0.7,0.8)$, $A_2 = (0.2,0.5,0.9)$ and $A_3 = (0.1,0.6,0.8)$ as shown in Figure 4.

TABLE 2: Ranking Results of Example 4

Index	Fuzzy numbers	Index value	Ranking results
Proposed index	$E_{SS}(A_1, A_2)$, $E_{SS}(A_2, A_1)$	$0.5+0.045\beta$, $0.784-0.06\beta$	$A_1 \prec A_3 \prec A_2$, $\beta \in [0,1]$
	$E_{SS}(A_2, A_3)$, $E_{SS}(A_3, A_2)$	0.810, 0.765	
	$E_{SS}(A_1, A_3)$, $E_{SS}(A_3, A_1)$	$0.538+0.033\beta$, $0.833-0.055\beta$	
Abbasbandy and Hajjari (2009)	A_1	0.5250	$A_2 \prec A_1 \prec A_3$
	A_2	0.5084	
	A_3	0.5750	
Asady and Zendehnam (2007)	A_1	0.475	$A_1 \prec A_2 \approx A_3$
	A_2	0.525	
	A_3	0.525	
Barkhordary *et al.* (2007)	A_1	-0.0136	$A_1 \prec A_2 \prec A_3$
	A_2	-0.0045	
	A_3	-0.0091	
Abbasbandy and Asady (2006) $p = 1$	A_1	0.95	$A_1 \prec A_2 \approx A_3$
	A_2	1.05	
	A_3	1.05	
Chu and Tsao (2002)	A_1	0.2440	$A_1 \prec A_3 \prec A_2$
	A_2	0.2624	
	A_3	0.2619	
Yao and Wu (2000)	A_1	0.475	$A_1 \prec A_2 \approx A_3$
	A_2	0.525	
	A_3	0.525	
Cheng (1998)	A_1	0.7106	$A_1 \prec A_3 \prec A_2$
	A_2	0.7256	
	A_3	0.7241	
Chen (1985)	A_1	0.52	$A_1 \prec A_2 \prec A_3$
	A_2	0.57	
	A_3	0.625	

The ranking values of the proposed method are shown in Table 2. Thus, the ranking order of the fuzzy numbers is $A_1 \prec A_3 \prec A_2$ regardless of the decision makers' perspective. However, Yao and Wu's (2000), Abbasbandy and Asady's (2006) and, Asady and Zendehnam's (2007) indices produce the ranking as $A_1 \prec A_2 \approx A_3$, which cannot discriminate the ranking of A_2 and A_3. Obviously, the results obtained by Yao and Wu's (2000), Abbasbandy and Asady's (2006) and, Asady and Zendehnam's (2007) are unreasonable. Moreover, Chen's (1985) and Barkhordary *et al.*'s (2007) methods produce the ranking as $A_1 \prec A_2 \prec A_3$, while Abbasbandy and Hajjari (2009) rank the fuzzy numbers as $A_2 \prec A_1 \prec A_3$. Other ranking methods such as Cheng's (1998) distance and Chu and Tsao's (2002) produce similar ranking results with the proposed method.

Example 5

Consider the data used in Wang *et al.* (2009), i.e., a triangular fuzzy number $A_1 = (1,2,5)$ and a fuzzy number $A_2 = (1,2,2,4)$ as shown in Figure 5.

The membership function of A_2 is defined as

$$\mu_{A_2}(x) = \begin{cases} \sqrt{1-(x-2)^2} & ,[1,2] \\ \sqrt{1-\frac{1}{4}(x-2)^2} & ,[2,4]. \\ 0 & ,\text{else} \end{cases}$$

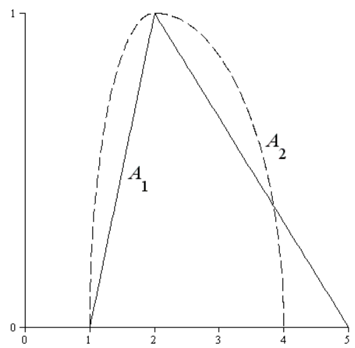

Figure 5: Fuzzy Numbers in Example 5

Some of the existing ranking methods such as Chen and Chen (2003), Chen and Chen (2007) and Chen and Chen (2009) can only rank trapezoidal fuzzy numbers and, thus, fail to rank the fuzzy numbers A_1 and A_2. By using the proposed method, we have $E_{SS}(A_1, A_2) = 0.806 - 0.008\beta$ and

$E_{SS}(A_2, A_1) = 0.660 + 0.029\beta$. Therefore, the ranking order is $A_1 \succ A_2$ regardless of the decision makers' perspective, as shown in Table 3. Based on Table 3, Deng *et al.*'s (2006) index produces the ranking order as $A_1 \prec A_2$ which is unreasonable. The ranking result of the proposed method is consistent with human intuition and other methods in Table 3.

Table 3: Ranking Results of Example 5

Index	**Fuzzy numbers**	**Index value**	**Ranking results**
Proposed index	$E_{SS}(A_1, A_2)$	$0.806 - 0.008\beta$	$A_1 \succ A_2$, $\beta \in [0,1]$
	$E_{SS}(A_2, A_1)$	$0.660 + 0.029\beta$	
Chen and Chen (2003)	A_1	3.162	*
	A_2	*	
Chen and Chen (2007)	A_1	*	*
	A_2	*	
Chen and Chen (2009)	A_1	0.371	*
	A_2	*	
Nejad and Mashinchi (2011)	A_1	0.274	$A_1 \succ A_2$
	A_2	0.190	
Wang *et al.* (2009)	A_1	0.2154	$A_1 \succ A_2$
	A_2	0	
Asady and Zendehnam (2007)	A_1	2.5	$A_1 \succ A_2$
	A_2	2.360	
Deng *et al.* (2006)	A_1	1.143	$A_1 \prec A_2$
	A_2	2.045	
Chu and Tsao (2002)	A_1	1.245	$A_1 \succ A_2$
	A_2	1.182	
Cheng (1998)	A_1	2.717	$A_1 \succ A_2$
	A_2	2.473	
Setnes and Cross (1997)	A_1	0.890	$A_1 \succ A_2$
	A_2	0.806	

'*': the ranking method cannot calculate the ranking value

6. CONCLUSION

This paper presents a new method for ranking fuzzy numbers using Sokal and Sneath index and Hurwicz criterion. The new method takes into consideration all types of decision makers' perspective which is crucial in solving decision-making problems. The proposed method can overcome certain shortcomings that exist in the previous ranking methods such as can rank both non-normal and general types of fuzzy numbers, and can discriminate the ranking of fuzzy numbers having the same mode and symmetric spreads which fails to be ranked by the previous ones. Besides, the results of the proposed method are also consistent with human intuition and most of other previous methods.

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