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On Some Inequalities of Hermite-Hadamard Type for *M*-Convex Functions

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ABSTRACT

In this paper, we attain some new upper bounds of the left hand side of the Hermite-Hadamard type inequalities for m-convex functions. Some applications to special means of positive real numbers are given.

Keywords: Hermite-Hadamard inequalities, m-convex function, Hölder inequality.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard inequalities:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subseteq \Re \to \Re$ is a convex function on the subinterval I of real numbers and $a, b \in I$ with $a < b$. For several recent results, generalization concerning Hermite-Hadamard inequalities. Throughout of this paper we consider a real interval $I \subset \Re$ and with I° we denote the interior of I.

In Toader (1984) defined the concept of *m*-convexity as the following:

Definition 1.1

A function $f:[0,b]\to\Re$, $b>0$, is said to be *m*-convex, where $m\in[0,1]$, if for all $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y).$$

In Mihesan (1993) defined the class of (α,m) -convex functions as the following:

Definition 1.2

A function $f:[0,b]\to\Re$, $b>0$, is said to be (α,m) -convex, where $(\alpha,m)\in[0,1]^2$, if for any $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y).$$

It is obvious that if $\alpha=1$, then (α,m) -convex means *m*-convex. For recent results and generalizations concerning *m*-convex and (α,m) -convex functions, see the references therein.

Definition 1.3

A function $f:I\subset[0,\infty)\to[0,\infty)$, is said to be *s*-convex on *I*, if the inequality $f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$ holds for all $x,y\in I$ and $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$.

Alomari *et al.* (2010) discussed the following results to the left hand side of the Hermite-Hadamard inequalities for *s*-convex functions:

Theorem 1.4

[Alomari *et al.* (2011), Theorem 2.2]. Let $f:I\subset[0,\infty)\to\Re$ be a differentiable mapping on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4(s+1)(s+2)}\left[|f'(a)|+|f'(b)|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|\right] \tag{2}$$

$$\leq \frac{(2^{2-s}+1)(b-a)}{4(s+1)(s+2)}\left[|f'(a)|+|f'(b)|\right]. \tag{3}$$

Theorem 1.5

[Alomari *et al.* (2011), Theorem 2.3] Let $f: I\subset[0,\infty)\rightarrow\Re$ be a differentiable mapping on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)},(p>1)$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}\times$$

$$\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right)^{\frac{1}{q}}\right. \tag{4}$$

$$\left.+\left(\left(2^{1-s}+s+1\right)|f'(b)|^q+2^{1-s}|f'(a)|^q\right)^{\frac{1}{q}}\right]$$

Theorem 1.6

[Alomari *et al.* (2011), Theorem 2.4] Let $f: I\subset[0,\infty)\rightarrow\Re$ be a differentiable mapping on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q,(q\geq1)$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\times$$

$$\left[\left\{ (2^{1-s}+1)\,|\,f'(a)\,|^{q} + 2^{1-s}\,|\,f'(b)\,|^{q} \right\}^{\frac{1}{q}} \right.$$

$$\left. + \left\{ (2^{1-s}+1)\,|\,f'(b)\,|^{q} + 2^{1-s}\,|\,f'(a)\,|^{q} \right\}^{\frac{1}{q}} \right]. \tag{5}$$

Dragomir and Agarwal (1998) established the following result connected with the right hand side of (1):

Theorem 1.7

[Dragomir and Agarwal (1998), Theorem 2.2] Let $f: I \subset \Re \to \Re$ be a differentiable mapping on I° where $a,b \in I$ with $a<b$. If $|\,f'\,|$ is convex on $[a,b]$, then the following inequality holds

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{b-a}{8}\left(|\,f'(a)| + |\,f'(b)\,| \right). \tag{6}$$

In Bakula et al. (2008), Dragomir (1993) and Eftekhari (2012) have been discussed results related to the right hand side of inequalities in (1), for *m*-convex functions. In this paper we establish some new inequalities for the left hand side of (1), for *m*-convex functions.

2. NEW INEQUALITIES FOR *m*-CONVEX FUNCTIONS

In order to prove our main results we need an auxiliary lemma.

Lemma 2.1

Let $f:[a,b] \to \Re$ be a differentiable function on (a,b), where $a,b \in \Re$ with $a<b$. If $f' \in L^1[a,b]$, then the following equality holds

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \tag{7}$$

$$= \frac{b-a}{4}\left[\int_0^1 (1-t)\, f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b \right)dt - \int_0^1 (1-t)\, f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a \right)dt \right].$$

Proof.

It suffices to note that by integrating by parts we have

$$I_1 = \int_0^1 (1-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt$$

$$= \frac{2(1-t)}{b-a} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\Bigg|_0^1 + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt$$

$$= \frac{-2}{a-b} f\left(\frac{a+b}{2}\right) + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt.$$

Setting $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ and $dx = \frac{a-b}{2}dt,$ which gives

$$I_1 = \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\varepsilon}{(a-b)^2}\int_a^{\frac{a+b}{2}} f(x)dx.$$

Similarly, we have

$$I_2 = \int_0^1 (1-t) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt$$

$$= \frac{-2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\varepsilon}{(a-b)^2}\int_{\frac{a+b}{2}}^{b} f(x)dx.$$

Therefore,

$$\frac{b-a}{4}(I_1 - I_2) = f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx,$$

which is required.

Remark 2.2

The above lemma is the same as Lemma 2.1 in Alomari *et al.* (2010), but for the left hand side of (1).

The following theorems provide new upper bounds for the left hand side of (1) for *m*-convex functions.

Theorem 2.3

Let $f: I \to \Re$, $I \subset [0,\infty)$ be a differentiable function on I° such that $f' \in L^1[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|$ is *m*-convex on $[a,b]$ for some fixed $m \in (0,1]$ then the following inequality holds

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \left(\frac{b-a}{12}\right) \times \left\{ |f'(a)| + |f'(b)| + \frac{m}{2}\left[\left| f'\left(\frac{a}{m}\right)\right| + \left| f'\left(\frac{b}{m}\right)\right| \right] \right\}. \tag{8}$$

Proof.

By Lemma 2.1, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \left(\frac{b-a}{4}\right) \times \tag{9}$$

$$\left[\int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt + \int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| dt. \right]$$

Since $|f'|$ is *m*-convex on $[a,b]$, then for any $t \in [0,1]$, we have

$$\left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| \le \frac{1+t}{2}|f'(a)| + m\left(\frac{1-t}{2}\right)\left| f'\left(\frac{b}{m}\right)\right|, \tag{10}$$

and similarly,

$$\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| \le \frac{1+t}{2}|f'(b)| + m\left(\frac{1-t}{2}\right)\left| f'\left(\frac{a}{m}\right)\right|. \tag{11}$$

Therefore, the inequalities (9), (10) and (11) imply

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{b-a}{4}\right)\times$$

$$\int_0^1\left[\left(\frac{1-t^2}{2}\right)(|f'(a)|+|f'(b)|)+m\frac{(1-t)^2}{2}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]dt$$

$$=\left(\frac{b-a}{4}\right)\left[\frac{|f'(a)|+|f'(b)|}{3}+\frac{m}{6}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]$$

$$=\left(\frac{b-a}{12}\right)\left[|f'(a)|+|f'(b)|+\frac{m}{2}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right].$$

Which completes the proof.

Remark 2.4

If in Theorem 2.3, one choose $m=1$, that is, $|f'|$ is convex on $[a,b]$, then we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{8}(|f'(a)|+|f'(b)|),$$

which is as the same as (6), but for the left hand side of (1).

Theorem 2.5

Suppose that all the assumptions of Theorem 2.3 are satisfied. Then the following inequality holds

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{12}\left[\frac{|f'(a)|+|f'(b)|}{2}+2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right].\quad(12)$$

Proof.

Since $|f'|$ is m-convex on $[a,b]$ and $\frac{1+t}{2}a+\frac{1-t}{2}b=ta+(1-t)\frac{a+b}{2}$, $\frac{1+t}{2}b+\frac{1-t}{2}a=tb+(1-t)\frac{a+b}{2}$, then for any $t\in[0,1]$, we have

$$\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\leq t\,|f'(a)|+m(1-t)\left|f'\left(\frac{a+b}{2m}\right)\right|,$$

and

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\leq t\,|f'(b)|+m(1-t)\left|f'\left(\frac{a+b}{2m}\right)\right|.$$

Therefore, by Lemma 2.1, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{b-a}{4}\right)\times$$

$$\int_0^1\left[t(1-t)(|f'(a)|+|f'(b)|)+2m(1-t)^2\left|f'\left(\frac{a+b}{2m}\right)\right|\right]dt$$

$$=\frac{b-a}{4}\left[\frac{|f'(a)|+|f'(b)|}{6}+\frac{2m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|\right]$$

$$=\frac{b-a}{12}\left[\frac{|f'(a)|+|f'(b)|}{2}+2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right],$$

which is (12).

Theorem 2.6

Let $f:I\to\Re$, $I\subset[0,\infty)$ be a differentiable function on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$ and $q>1$ then the following inequalities hold

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left\{\left[|f'(a)|^q+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}+\left[|f'(b)|^q+\frac{m}{2}\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}$$

$$=\left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[|f'(a)|+|f'(b)|+\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]. \quad (13)$$

Proof.

Since $|f'|^q$ is *m*-convex on $[a,b]$, we have

$$\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\leq\left(\frac{1+t}{2}\right)|f'(a)|^q+m\left(\frac{1-t}{2}\right)\left|f'\left(\frac{b}{m}\right)\right|^q, \quad (14)$$

and

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q\leq\left(\frac{1+t}{2}\right)|f'(b)|^q+m\left(\frac{1-t}{2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q. \quad (15)$$

Using inequalities (14), (15) and well known Hölder's inequality for $p=\frac{q}{q-1}$ and $q>1$, we have

$$\left[\int_0^1(1-t)\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right]$$

$$=\int_0^1(1-t)^{1-\frac{1}{q}}(1-t)^{\frac{1}{q}}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt$$

$$\leq\left(\int_0^1(1-t)dt\right)^{\frac{q-1}{q}}\left(\int_0^1(1-t)\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left[\int_0^1 \left(\frac{(1-t^2)}{2} | f'(a)|^q + m\frac{(1-t)^2}{2}\left| f'\left(\frac{b}{m}\right)\right|^q\right) dt\right]^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\frac{1}{3} | f'(a)|^q + \frac{m}{6}\left| f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \tag{16}$$

and similarly

$$\int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| dt \le \left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\frac{1}{3} | f'(b)|^q + \frac{m}{6}\left| f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

Now using inequality (9), we get

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}} \left\{\left[\left| f'(a)\right|^q + \frac{m}{2}\left| f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} + \left[| f'(b)|^q + \frac{m}{2}\left| f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Using the fact that

$$\sum_{i=1}^n (a_i + b_i)^r \le \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r \tag{17}$$

for $0 < r < 1$, $a_1, a_2, \ldots, a_n \ge 0$ and $b_1, b_2, \ldots, b_n \ge 0$, we obtain (13).

Theorem 2.7

Suppose that all the assumptions of Theorem 2.6 are satisfied. Then the following inequalities hold

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}} \times$$

$$\left\{\left[\frac{|f'(a)|^q}{2}+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{\frac{1}{q}}+\left[\frac{|f'(b)|^q}{2}+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{\frac{1}{q}}\right\}$$

$$\le \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|+|f'(b)|}{2^{\frac{1}{q}}}+2m^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2m}\right)\right|\right].$$

Proof.

We proceed as in the proof of Theorem 2.6, but instead of inequalities (14) and (15), we use the following inequalities:

$$\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q \le t\,|f'(a)|^q + m(1-t)\left|f'\left(\frac{a+b}{2m}\right)\right|^q, \qquad (18)$$

and

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q \le t\,|f'(b)|^q + m(1-t)\left|f'\left(\frac{a+b}{2m}\right)\right|^q. \qquad (19)$$

So inequalities (9), (16), (17), (18) and (19) imply

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \left(\frac{b-a}{4}\right)\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \times$$

$$\left\{\left(\int_0^1\left[t(1-t)\,|f'(a)|^q + m(1-t)^2\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left[t(1-t)\,|f'(b)|^q + m(1-t)^2\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$= \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}} \left\{ \left[\frac{|f'(a)|^q}{2} + m\left| f'\left(\frac{a+b}{2m}\right) \right|^q \right]^{\frac{1}{q}} + \left[\frac{|f'(b)|^q}{2} + m\left| f'\left(\frac{a+b}{2m}\right) \right|^q \right]^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}} \left[\frac{|f'(a)|+|f'(b)|}{2^{\frac{1}{q}}} + 2m^{\frac{1}{q}}\left| f'\left(\frac{a+b}{2m}\right)\right| \right],$$

which completes the proof.

Theorem 2.8

Suppose that all the assumptions of Theorem 2.6 are satisfied. Then the following inequalities hold

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \left(\frac{b-a}{16}\right)\left(\frac{4q-4}{2q-1}\right)^{\frac{q-1}{q}} \times$$

$$\left\{ \left[3|f'(a)|^q + m\left| f'\left(\frac{b}{m}\right)\right|^q \right]^{\frac{1}{q}} + \left[3|f'(b)|^q + m\left| f'\left(\frac{a}{m}\right)\right|^q \right]^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{b-a}{16}\right)\left(\frac{4q-4}{2q-1}\right)^{\frac{q-1}{q}} \left[3^{\frac{1}{q}}\left(|f'(a)|+|f'(b)|\right) + m^{\frac{1}{q}}\left(\left| f'\left(\frac{a}{m}\right)\right| + \left| f'\left(\frac{b}{m}\right)\right| \right) \right].$$

Proof.

Using inequalities (14), (15) and (17) and well known Hölder's inequality for $p = \frac{q}{q-1}$ and $q > 1$, we have

$$\begin{aligned}
&\int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| dt \\
&\le \left(\int_0^1 (1-t)^p\, dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (20)\\
&\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left[\int_0^1 \left(\left(\frac{1+t}{2}\right)| f'(a)|^q + m\left(\frac{1-t}{2}\right)\left| f'\left(\frac{b}{m}\right)\right|^q\right) dt\right]^{\frac{1}{q}} \\
&= \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{3}{4}| f'(a)|^q + \frac{m}{4}\left| f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{4}\right)^{\frac{1}{q}} \left(3^{\frac{1}{q}}\, | f'(a)| + m^{\frac{1}{q}}\left| f'\left(\frac{b}{m}\right)\right|\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 (1-t)\left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right| dt \\
&\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{4}\right)^{\frac{1}{q}} \left(3\ \, | f'(b)|^q + m\left| f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}, \qquad (21)\\
&\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{1}{4}\right)^{\frac{1}{q}} \left(3^{\frac{1}{q}}\, | f'(b)| + m^{\frac{1}{q}}\left| f'\left(\frac{a}{m}\right)\right|\right).
\end{aligned}$$

Thus the combination of (9), (20) and (21) imply desired inequalities.

If in Theorem 2.8, we use (9), (17), (18), (19) and (20), then we get the following theorem.

Theorem 2.9

Suppose that all the assumptions of Theorem 2.6 are satisfied. Then the following inequalities hold

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{b-a}{8}\right)\left(\frac{2q-2}{2q-1}\right)^{\frac{q-1}{q}}\times\sum$$

$$\left\{\left[|f'(a)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{\frac{1}{q}}+\left[|f'(b)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{b-a}{8}\right)\left(\frac{2q-2}{2q-1}\right)^{\frac{q-1}{q}}\left[\left(|f'(a)|+|f'(b)|\right)+2m^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2m}\right)\right|\right].$$

Remark 2.10

Since we have

$$\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}<1,\qquad q\in(1,\infty).$$

We can get new upper bounds in the previous theorems.

3. APPLICATIONS TO SPECIAL MEANS

Now by using the results of Section 2, we give some applications to special means of positive real numbers.

(1) The arithmetic mean: $A(a,b)=\dfrac{a+b}{2}$, $a,b\in\Re$, $a,b>0.$

(2) The logarithmic mean: $L(a,b)=\dfrac{b-a}{\ln b-\ln a}$, $a,b\in\Re$, $a\neq b$, $a,b>0.$

(3) The generalized logarithmic mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}},\quad n\in\Re\setminus\{-1,0\},\quad a,b\in[0,\infty),\quad a\neq b.$$

The following propositions hold.

Proposition 3.1

Let $n \in (-\infty, 0) \bigcup [1, \infty) \setminus \{-1\}$ and $[a,b] \subset (0,\infty)$. Then we have the following inequality

$$\left| A^n(a,b) - L_n^n(a,b) \right| \le n\left(\frac{b-a}{12}\right)\left[A(a^{n-1}, b^{n-1}) + 2A^{n-1}(a,b) \right].$$

Proof.

The assertion follows from Theorem 2.5 for $f(x) = x^n$ and n as specified above and $m = 1$.

Proposition 3.2

Let $n \in (-\infty, 0) \bigcup [1, \infty) \setminus \{-1\}$ and $[a,b] \subset (0,\infty)$ and $q > 1$. Then we have the following inequality:

$$\left| A^n(a,b) - L_n^n(a,b) \right| \le n\left(\frac{b-a}{4}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[2^{-\frac{1}{q}} A(a^{n-1}, b^{n-1}) + A^{n-1}(a,b) \right].$$

Proof.

The assertion follows from Theorem 2.7 for $f(x) = x^n$ and n as specified and $m = 1$.

Proposition 3.3

Let $q > 1$ and $[a,b] \subset (0,\infty)$. Then we have the following inequality

$$\left| A^{-1}(a,b) - L^{-1}(a,b) \right| \le \left(\frac{b-a}{8}\right)\left(\frac{4q-4}{2q-1}\right)^{\frac{q-1}{q}} (3^{\frac{1}{q}} + 1) A(a^{-2}, b^{-2}).$$

Proof.

The assertion follows from Theorem 2.8 for $f(x) = \dfrac{1}{x}$ and $m = 1$.

4. CONCLUSION

If in Theorem 2.5, we put $m=1$, then inequalities (2) and (3) for $s=1$ follow and for $m=1$ in Theorem 2.8, inequality (4) is the same as inequality (4) for $s=1$. Also, in Theorem 2.6, if we set $m=1$, then inequality (5) for $s=1$ follows. In Eftekhari (2012) obtained some upper bound to the left hand side of (1) for (α,m)-convex functions, if we set $\alpha=1$, then the results of this paper attain.

Theorems 2.6 - 2.9 imply the following remark.

Remark 4.1

From Theorems 2.6 - 2.9, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \min\{E_1, E_2, E_3, E_4\},$$

where

$$E_1 = \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[|f'(a)| + |f'(b)| + \left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|\right)\right],$$

$$E_2 = \left(\frac{b-a}{8}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[\frac{|f'(a)| + |f'(b)|}{2^{\frac{1}{q}}} + 2m^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2m}\right)\right|\right],$$

$$E_3 = \left(\frac{b-a}{16}\right)\left(\frac{4q-4}{2q-1}\right)^{\frac{q-1}{q}}\left[3^{\frac{1}{q}}\left(|f'(a)| + |f'(b)|\right) + m^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|\right)\right],$$

$$E_4 = \left(\frac{b-a}{8}\right)\left(\frac{2q-2}{2q-1}\right)^{\frac{q-1}{q}}\left[\left(|f'(a)| + |f'(b)|\right) + 2m^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2m}\right)\right|\right].$$

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