Malaysian Journal of Mathematical Sciences 8(2): 255-264 (2014)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Power Geometry for Finding Periodic Solutions in One System of ODE

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ABSTRACT

In a vicinity of a stationary solution we consider a real analytic system of ODE of order four, depending on a small parameter. We look for families of periodic solutions which contract to the stationary solution, when the parameter tends to zero. We apply the general method of power geometry for the study of complex bifurcations for local resolutions of singularities.

Keywords: Stationery solution, periodic solution, singularity, method of power geometry, complicated bifurcations, Newton polyhedral, truncated system, normal form, local resolutions, power transformations.

1. INTRODUCTION

First of all, we bring the system to a normal form in a vicinity of a fixed point, then we compute the set A containing all the families of periodic solutions that contract to this fixed point. These families can be written as asymptotic power series in a small parameter. To obtain the first few terms of these series from the normal form, we single out the first approximation of the system (truncated system) and study it in detail. In the nondegenerate case it is the truncated system that determines character of the bifurcations and their asymptotic. The higher terms in the normal form allow one to make the asymptotic expansion of the family more precise. Thus, the computation of these families of periodic solutions is performed over the coefficients of the terms of ire normal form. For concrete systems, the computation of the coefficients of terms in the normal form can be made only up to terms of some finite degree. In this case it is important to compute all coefficients of the terms of the lowest degree (that appear in the truncated system).

2. DESCRIPTION OF THE METHOD

We consider a real analytic system whose expression in complex conjugate coordinates is

$$\begin{aligned} dy_1/dt &= a(\varepsilon)y_1 + f_1(\varepsilon, y_1, y_2, \bar{y}_1, \bar{y}_2), \\ dy_2/dt &= a(\varepsilon)y_2 + f_2(\varepsilon, y_1, y_2, \bar{y}_1, \bar{y}_2), \end{aligned} \tag{1}$$

and the corresponding complex conjugate equations. We assume that $a(0) = i = \sqrt{-1}$ and the functions f_1 and f_2 are expanded into power series without any free and linear terms in $y_j, \bar{y}_j, j = 1, 2$. We look for families of periodic solutions for (1), which contract to the stationary point $y_1 = y_2 = \bar{y}_1 = \bar{y}_2 = 0$, when the small parameter ε tends to zero (see Bruno and Soleev (1992)). Then the normal form of the system (1) is as follows

$$\begin{aligned} du_1/dt &= a(\varepsilon)u_1 + \Phi_1(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2), \\ du_2/dt &= a(\varepsilon)u_2 + \Phi_2(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) \end{aligned} \tag{2}$$

and the corresponding conjugate equations, where

$$\begin{gathered} a(\varepsilon) = i + d_1\varepsilon + \cdots \\ \Phi_2(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) = \textstyle\sum_Q a_{jQ} u_1^{q_1} u_2^{q_2} \bar{u}_1^{q_3} \bar{u}_2^{q_4} \end{gathered} \tag{3}$$

with $Q = (q_1, q_2\ , q_3, q_4)$ and $q_1 + q_2 - q_3 - q_4 = 1$.

For small $|y_1|, |y_2|$ and ε, all desired families of periodic solutions of the system (1) are in the set Á (Soleev (1983)) which is determined from the normal form (2) by the system of four equations

$$\begin{aligned} a(\varepsilon)u_j + \Phi_j(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) &= a(0)\alpha u_j, \\ \bar{a}(\varepsilon)\bar{u}_1 + \bar{\Phi}_1(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) &= \bar{a}(0)\alpha\bar{u}_j,\ (j = 1,2) \end{aligned} \tag{4}$$

where α is a parameter. Eliminating α, we obtain a system of three analytical equations in four independent variables

$$\begin{aligned} g_1 &\stackrel{\text{def}}{=} u_2\Phi_1 - u_1\Phi_2 = 0, \\ g_2 &\stackrel{\text{def}}{=} \bar{g}_1 = \bar{u}_2\bar{\Phi}_1 - \bar{u}_1\bar{\Phi}_2 = 0, \\ g_3 &\stackrel{\text{def}}{=} \big(a(\varepsilon) + \bar{a}(\varepsilon)\big)u_2\bar{u}_2 + \bar{u}_2\Phi_2 + u_2\bar{\Phi}_2 = 0. \end{aligned} \tag{5}$$

In a small vicinity near the stationary point $u_1 = u_2 = \bar{u}_1 = \bar{u}_2 = 0$, the set of solutions of system (5) have branches. We shall find all these branches by means of the method developed in Bruno (1993, 2000). Taking into account the first terms of the power series (3), we find the supports of the polynomials g, for the system (5):

$$D(g_1) = \{Q_1^1 = (2,1,1,0), Q_2^1 = (1,2,0,1), Q_3^1 = (1,2,1,0), Q_4^1 = (2,1,0,1),$$
$$Q_5^1 = (0,3,0,1), Q_6^1 = (0,3,1,0), Q_7^1 = (3,0,1,0), Q_8^1 = (3,0,0,1), \ldots\};$$

$$D(g_2) = \{Q_1^2 = (0,1,0,2), Q_2^2 = (0,1,1,2), Q_3^2 = (1,0,1,2), Q_4^2 = (0,1,2,1),$$
$$Q_5^2 = (0,1,0,3), Q_6^2 = (1,0,0,3), Q_7^2 = (1,0,3,0), Q_8^2 = (0,1,3,0), \ldots\};$$

$$D(g_3) = \{Q_1^3 = (1,0,0,1), Q_2^3 = (2,0,1,1), Q_3^3 = (1,1,0,2), Q_4^3 = (1,1,1,1),$$
$$Q_5^3 = (0,,2,0,2), Q_6^3 = (0,2,1,1), Q_7^3 = (2,0,2,0), Q_8^3 = (1,1,2,0), \ldots\}.$$

For the supports $D(g_i)$ obtained above, we can compute the corresponding Newton polyhedral and normal cones (see Bruno (1993)). The computation shows that the system (5) has only one truncation whose normal cone is $\mathrm{IR}_+\Omega$, where $\Omega = (-1,-1,-1,-1)$, $(\mathrm{IR}_+ = \{t\mathrm{IR}, t \geq 0\})$. The truncated subsystem associated with the cone $\mathrm{IR}_+\Omega$ consists of

$$\hat{g}_1 \overset{\text{def}}{=} b_1 u_1^2 u_2 \bar{u}_1 + b_2 u_1 u_2^2 \bar{u}_2 + b_3 u_1 u_2^2 \bar{u}_1 + b_4 u_1^2 u_2 \bar{u}_2 + b_5 u_2^3 \bar{u}_2 + b_6 u_2^3 \bar{u}_1 - b_7 u_1^3 \bar{u}_1 - b_8 u_1^3 \bar{u}_2 = 0. \tag{6}$$

and its conjugate equation.

Considering the vectors $T_1 = Q_7^1 - Q_1^1 = (1,-1,0,0), T_2 = Q_7^2 - Q_1^2 = (0,0,1,-1)$ and $T_3 = Q_3^3 - Q_1^3 = (0,1,0,1)$, we construct a unimodular matrix (by adding on an extra vector $T_4 = (1,0,0,0)$).

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \textit{with univers } \alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

The power transformations corresponding to these matrices are

$$\left.\begin{aligned} u_1 &= u_1, \\ z &= u_1 u_2^{-1}, \\ \bar{z} &= u_1 \bar{u}_2^{-1}, \\ r &= u_2 \bar{u}_2, \end{aligned}\right\} \qquad \left.\begin{aligned} u_1 &= u_1, \\ u_2 &= u_1 z^{-1}, \\ \bar{u}_1 &= u_1^{-1} z\bar{z}r, \\ \bar{u}_2 &= u_1^{-1} zr \end{aligned}\right\}. \tag{7}$$

Under the power transformation (7) and the reduction by u_1u_2 in the first equation, by $\bar{u}_1\bar{u}_2$ in the second one and by $u_2u\bar{u}_2$ in the third one, the system (5) can be converted into

$$\begin{aligned} G_1 &\overset{\text{def}}{=} \psi_1(\varepsilon, z, \bar{z}, r) - \psi_2(\varepsilon, z, \bar{z}, r) = 0, \\ G_2 &\overset{\text{def}}{=} \overline{\psi_1} - \overline{\psi_2} = 0, \\ G_3 &\overset{\text{def}}{=} \big(a(\varepsilon) + \bar{a}(\varepsilon)\big) + (\psi_2(\varepsilon, z, \bar{z}, r)\ -\ \overline{\psi_2}(\varepsilon, z, \bar{z}, r) = 0, \end{aligned} \tag{8}$$

where

$$\Phi_2(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) = u_j \psi_2(\varepsilon, z, \bar{z}, r).$$

After the reduction by $u_1u_2z^{-1}r$ in the first equation and by $\bar{u}_1\bar{u}_2\bar{z}^{-1}r$ in the second one, the truncated system (6) is translated into

$$\overline{G_1} \overset{\text{def}}{=} b_1z^2\bar{z} + b_2z + b_3z\bar{z} + b_4z^2 + b_5 + b_6\bar{z} - b_7z^3\bar{z} - b_8z^3 = 0 \tag{9}$$

and its conjugate equation. From the first equation of system (9) we find

$$\bar{z} = \frac{b_5 + b_2z + b_4z^2 - b_8z^3}{b_7z^3 - b_1z^2 - b_3z - b_6}. \tag{10}$$

If we substitute $\bar{z}$ into the second equation of system (9) we obtain an algebraic equation of degree 10 in z. Consequently, system (9) has ten complex roots $(z_0, \bar{z}_1)$, but not for all of them $\bar{z}_0 = \bar{z}_1$.

Theorem 1.

There exists such system (1), that the system (9) has 10 simple roots $(z_0, \bar{z}_0)$, i.e. they are 10 real coordinates.

Proof.

Let for system (9) $b_3 = b_4 = b_5 = b_7 = 0.$ Then (10) is

$$\bar{z} = -z\frac{(b_2 - b_8 z^2)}{(b_6 + b_1 z^2)}. \tag{11}$$

Denote

$$x = \frac{b_2 - b_8 z^2}{b_6 + b_1 z^2}. \tag{12}$$

Then

$$z^2 = \frac{b_2 - b_6 x}{b_8 + b_1 x} \tag{13}$$

and the equation (11) becomes

$$\bar{z}/z = x. \tag{14}$$

From this we see that

$$|x|=1 \tag{15}$$

for solutions which are interesting for us i.e. $x\bar{x} = 1$. By squaring both sides of (14) we obtain the equation

$$\bar{z}^2 = x^2 z^2.$$

According to (13) and ($\overline{13}$) after the change $\bar{x} = 1/x$, it turns into

$$\frac{\bar{b}_2 x - \bar{b}_6}{\bar{b}_8 x + \bar{b}_1} = x^2\frac{b_2 - b_6 x}{b_8 + b_1 x},$$

which is equivalent to an equation of degree 4. We need solutions that satisfy the relations (14) and (15). To make them more explicit, we multiply the equation (11) $\bar{z} = -xzbyz$. Then according to (13) for $x\bar{x} = 1$ we have

$$z\bar{z} = -z^2 x = -x\frac{b_2 - b_6 x)}{b_1 x + b_8} = -\frac{(b_2 - b_6 x)(\bar{b}_1 + \bar{b}_8 x)}{(b_1 x + b_8)(\bar{b}_1 \bar{x} + \bar{b}_8)}.$$

Since $Imz\bar{z} = 0$ and $Rez\bar{z} > 0$, we obtain

$$Im(b_2 - b_6 x)\left(b\overline{b_8}x + \overline{b_1}\right) = 0. \quad |x|=1 \tag{16}$$

$$Re(b_2 - b_6 x)(\overline{b_8}x + \overline{b_1}) < 0. \tag{17}$$

The equations (16) are two quadratic equations with respect to *Re(x)* and *Im(x)* . After the elimination one of them, we obtain equation of degree 4 such that from its roots we can choose only those that satisfy the inequality (17).

Now we prove that there exists such system (16), (17) with 4 solutions. For that in the complex plain we consider the points of intersection of the circle $|x|=1$ and the hyperbola

$$Im(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) = 0 .$$

Here the points $x = b_2/b_6$ and $x = -\bar{b}_1/\bar{b}_8$ lie in this hyperbola and are used for boundary of those its points that satisfy the inequality (17), whereas the point

$$x = ((b_2/b_6) - (\bar{b}_1/\bar{b}_8))/2$$

in the center of the hyperbola. For simplicity, we restrict our self with the case $b_6 = b_8 = 1.$ Then

$$(x - b_2)(x + \bar{b}_1) = x - (\frac{b_2 - \bar{b}_1}{2})^2 - (\frac{b_2 + \bar{b}_1}{2})^2$$

and

$$Re(x - b_2)(x + \bar{b}_1) = [Re(x - \frac{b_2 - \bar{b}_1}{2})]^2 - [Im(x - \frac{b_2 - \bar{b}_1}{2})]^2$$
$$+1/4[Re(b_2 + \bar{b}_1)]^2 - 1/4[Im(b_2 + \bar{b}_1)]^2.$$

$$Im(x - b_2)(x + \bar{b}_1) = Re(x - \frac{b_2 - \bar{b}_1}{2})Im(x - \frac{b_2 - \bar{b}_1}{2})$$
$$+1/4Re(b_2 + \bar{b}_1)Im(b_2 + \bar{b}_1)$$

On the hyperbola

$$Im(x - b_2)(x + \bar{b}_1) = 0$$

the inequality

$$Re(x - b_2)(x + \bar{b}_1) < 0$$

means that the Rex lies on the interval

$$J = (min[Reb_2 - Reb_1], \quad max[Reb_2 - Reb_1]),$$

if $Re(b_2) \neq Re(b_1)$.

Further, we restrict ourselves to the case $Re(b_2) \neq Re(b_1)$ and $Im(b_2) = Im(b_1)$.. Then the first equation in (16) defines two perpendicular lines

$$Re(x) = \frac{1}{2}Re(b_2) - \overline{b_1}, \qquad Im(x) = \frac{1}{2}Im(b_2) - \overline{b_1}. \tag{18}$$

The condition (17) is satisfied on the whole first line and in the interval J on the second line. Now we consider the case, when both lines (18) intersect the unit circle and both points b_2 and $-\overline{b}_1$ lie outside it, i.e.,

$$|Re(b_2) - Re(b_1)| < 2, |Im(b_2)| < 1, |b_2| > 1, |b_1| > 1.$$

Then the first line intersects the unit circle in two points and the interval J intersects it also in two points, i.e. we have 4 solutions of the system (16), (17).

According to (13), to each suitable value x^0 there corresponds two values $\pm z_0$.. Hence we have 8 different solutions $(z_0, \overline{z}_1)$ with $\overline{z}_1 = \overline{z}_0 \neq 0$ and ∞.

In addition the equation (9) has the root $z_0 = 0$ since $b_5 = 0$ and the root $z_0 = \infty$ since $b_7 = 0$. Evidently $\overline{z}_1 = \overline{z}_0$ for them. So the equation (9) has 10 roots with that property. This finishes the proof of the theorem.

Now we shall go back, and solve the system (8) with respect to four variables $z, \overline{z}, r, \varepsilon$.. For small ε and r solutions of system (8) belong to the vicinity of the point $(z_0, \overline{z}_0)$. We assume that the point $(z_0, \overline{z}_0)$ is the simple root of the system (9), i.e. in it the Jacobian

$$D(\hat{G}, \hat{\hat{G}}) / D(z, \bar{z}) \neq 0.$$

Then taking $z = z_0$, $\overline{z} = \overline{z}_0$, $r = 0$, $\varepsilon = 0$ and applying the Implicit Function Theorem we obtain the roots of the system (8) in the form of expansions

$$\begin{cases} z = z_0 + O(r), \\ \bar{z} = \overline{z_0} + O(r), \\ \varepsilon = -\frac{r}{2Red_1}\left[2Re\left(\widehat{\psi_1}(0, z_0, \overline{z_0}, 1)\right) + O(r)\right]. \end{cases} \tag{19}$$

Substituting these expansions into (7) we obtain

$$\begin{cases} u_2 = u_1\big(z_0 + O(r)\big)^{(-1)}, \\ \overline{u_1} = u_1^{-1} r\big(z_0\overline{z_0} + O(r)\big), \\ \overline{u_2} = u_1 r\big(z_0 + O(r)\big), \\ \varepsilon = -r\big(M + O(r)\big), \end{cases} \tag{20}$$

where $M = Re\big[\widehat{\psi_1}(0, z_0, \overline{z_0}, 1\big]/\, Re(d_1)$.

After substituting (20) into the sum (3), the first equation of system (2) implies

$$\frac{d\ln u_1}{dt} = a(-r(M + O(r))) + \psi^0(z_0, \bar{z}_0, r),$$

which yields $u_1 = Ce^{\theta t}$ where

$$\theta = a\Big(-r\big(M + O(r)\big)\Big) + \psi^0(z_0, \overline{z_0}, r), C = constant.$$

Consequently, from (20) we can obtain a family of periodic solutions of the system (2), corresponding to the roots (19) of the system

$$\begin{cases} u_2 = e^{\theta t}\big(z_0 + O(r)\big)^{(-1)}, \\ \overline{u_1} = e^{\theta t} r\big(z_0\overline{z_0} + O(r)\big), \\ \overline{u_2} = e^{\theta t} r\big(z_0 + O(r)\big), \\ \varepsilon = -r\big(M + O(r)\big), \end{cases} \tag{21}$$

As all solutions $(z_0, \bar{z}_0)$ obtained by the Theorem 1 are simple, so for corresponding series (20) and families of periodic solutions in form (21). So we have proved the following theorem.

Theorem 2.

There exist systems (1), in which 10 families of real periodic solutions bifurcate from the stationary point y=0, when e passes through zero.

If $(z_0, \bar{z}_0)$ is not a simple root of the system (9), we substitute

$$z = z_0 + \eta, \bar{z} = \bar{z}_0 + \bar{\eta}$$

into the system (8), which produces

$$\begin{aligned} H_1(\varepsilon, \eta, \bar{\eta}, r) &\stackrel{\text{def}}{=} G_1(\varepsilon, z_0 + \eta, \overline{z_0} + \overline{\eta_0}, r) = 0, \\ H_2(\varepsilon, \eta, \bar{\eta}, r) &\stackrel{\text{def}}{=} G_2(\varepsilon, z_0 + \eta, \overline{z_0} + \overline{\eta_0}, r) = 0, \\ H_2(\varepsilon, \eta, \bar{\eta}, r) &\stackrel{\text{def}}{=} G_2(\varepsilon, z_0 + \eta, \overline{z_0} + \overline{\eta_0}, r) = 0, \end{aligned} \tag{22}$$

To this system we apply the toroidal blowing up process used to pass from system (5) to system (8). In our case the singularity is concentrated at the point $\varepsilon = \eta = r = 0$. After the application of the procedure, the point will be blown up into a plane, and we must find several roots of a new truncated system. Sum of their multiplicities is exactly the multiplicity of the root $(z_0, \bar{z}_0)$. So each of new roots is simpler than the initial root. We can iterate this process until we obtain a nonsingular system. This way we can determine nil the components of the families of periodic solutions of system (2) which contract to the singular point (see Bruno (1999), Bruno and Soleev (1992) and Soleev (2005)).

3. CONCLUSION

Here we shown how works methods Power Geometry for the real system of ODEs of order four near a stationary pont, depending on a small parameter. The computations and investigations in this paper are based on two methods: a method introduced in Bruno (2000) to analyze complicated bifurcations and a method presented in Bruno (1993) to compute local resolutions of singularities. We have received that there exist systems (1), in which ten families of real periodic solutions bifurcate from the stationary point, when the parameter tends to zero. In the same manner, one can study periodic solutions of the Hamiltonian system with two degrees of freedom near a resonant periodic solution. Generally, bifurcations of periodic modes in resonant cases from Poiseuille flow, Couette flow and other flows were investigated by this way (see Bruno (1993) and Drazin (1992)).

ACKNOWLEDGEMENTS

Part of this work was carried out during the author's visit in University Putra Malaysia. The author would like to express his gratitude to researchers from Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research and Fundamental Research Grant Scheme (Project code is 01-12-10-989FR) for their support.

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