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Some Types of Spectral Distances between a Hypercube and its Complement and Line Graph

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ABSTRACT

Suppose M_1 and M_2 are two $n \times n$ matrices with eigenvalues $\lambda_1(M_j) \leq \lambda_2(M_j) \leq \cdots \leq \lambda_n(M_j)$, j = 1, 2. The spectral distance between M_1 and M_2 is defined as $\sigma(M_1, M_2) = \sum_{i=1}^{n} |\ \lambda_i(M_1) - \lambda_i(M_2)|$. In this paper, the Seidel, Laplacian, Signless Laplacian and Normalized Laplacian spectral distances of the hypercube and its complement, as well as the $k-$ iterated line graphs of hypercube and its complements are computed. Some results on the spectral distance double cover are also presented.

Keywords: Seidel spectral distance, Laplacian spectral distance, Normalized Laplacian spectral distance, Signless Laplacian spectral distance, Extended double cover of graph.

1. INTRODUCTION

In this section we recall some definitions that will be used in the paper. Let G be a simple graph with n vertices and m edges and $A(G) = (A_{ij})_{n\times n}$ be the adjacency matrix of G. The characteristic polynomial of the adjacency matrix $A(G)$, $P(G,\lambda) = P(A(G),\lambda) = \det(\lambda I - A(G))$ is called the characteristic polynomial of the graph G. The eigenvalues of the matrix $A(G)$ are called the eigenvalues of G. Suppose the eigenvalues of G are denoted $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$.The Seidel adjacency matrix of a graph G with adjacency matrix $A(G)$ is the matrix S defined by $S = J - I - 2A(G)$ (Cvetković, Doob and Sachs (1980)). The Laplacian matrix is defined by $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of G. The Laplacian eigenvalues of G are denoted by $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. The sigenless Laplacian eigenvalues of G are denoted $q_1(G) \leq q_2(G) \leq \cdots \leq q_n(G)$. The normalized Laplacian matrix $N(G) = (N_{ij})_{n\times n}$ is defined as (Chung (1997)):

$$N_{ij} = \begin{cases} 1 & if\ i = j\ and\ \deg(v_i) \neq 0 \\ \dfrac{-1}{\sqrt{\deg(v_i)\deg(vj)}} & if\ i \neq j\ and\ v_i\ is\ adjacent\ to\ v_j. \\ 0 & otherwise \end{cases}$$

The normalized Laplacian eigenvalues of G are denoted by $\delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G)$. The line graph $l(G)$ of G is the graph with the edge set of G as vertex set, where two vertices are adjacent if the corresponding edges of G have an endpoint in common.

The n −dimensional hypercube Q_n is the n −regular graphs whose vertices are the n −tuples with entries in $\{0,1\}$ and whose edges are the pairs of n −tuples that differ in exactly one position. It can easily seen that the number of vertices and edges of Q_n are 2^n and $n2^{n-1}$,respectively. The complement $\bar{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ and $uv \in e(\bar{G})$ if and only if uv not in $e(G)$. Let $\overline{Q_n}$ denote the complement of $n-$ dimensional hypercube Q_n (Chen (2012)). The Boolean lattice BL_n $(n \geq 1)$ is the graph whose vertex set is the set of all subsets of $\{1,2,\cdots,n\}$, where two subsets X and Y are adjacent if their symmetric difference has precisely one element (Bondy and Murty (1979), p. 9). The hypercube Q_n and Boolean lattice BL_n are isomorphic (Bondy and Murty (1979), p. 18). The following lemma is crucial throughout this paper.

Lemma 1. 1.

(i) (Cvetković, Doob and Sachs (1980). Let G be a r −regular graph on n vertices with eigenvalues $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G) = r$. Then the spectrum of $\bar{G}$ contains $n - r - 1$, $-1 - \lambda_1(G)$, ... , $-1 - \lambda_{n-1}(G)$.

(ii) (Brouwer and Haemers (2012). Let G be a r −regular graph on n vertices with $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G) = r$ .Then Seidel spectrum of G contains $n - 2r - 1, -1 - 2\lambda_1(G)$, ... , $-1 - 2\lambda_{n-1}(G)$.

(iii) (Cvetković, Doob and Sachs (1980). Let S be the Seidel matrix of the graph G, then Seidel matrix of the graph $\bar{G}$ is $\bar{S} = -S$.

(iv) (Brouwer and Haemers (2012), p. 4) If eigenvalues $\lambda_1(G),\ \lambda_2(G), \ldots, \lambda_n(G)$ are the eigenvalue of $r-$regular graph G, then $\mu_i(G) = r - \lambda_i(G)$ for $1 \leq i \leq n$ are the eigenvalues of $L(G)$.

(v) (Brouwer and Haemers (2012). The eigenvalues of $\overline{L(G)}$ are 0, $n - \mu_i(G)$ for $1 \leq i \leq n-1$.

(vi) (Brouwer and Haemers (2012). If $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ are the eigenvalue of $r-$ regular graph G, then $q_i(G) = r + \lambda_i(G)$, $1 \leq i \leq n$ are the eigenvalues of $Q(G)$.

(vii) The eigenvalues of $\overline{\mathrm{Q(G)}}$ are $2n - 2 - q_n(G)$, $n - 2 - q_i(G)$ for $1 \leq i \leq n-1$.

(viii) (Chung (1997). Let G be a $r-$regular graph. Then $\delta_i(G) = \frac{\mu_i(G)}{r}$ for $1 \leq i \leq n$.

(ix) The eigenvalues of $\overline{N(G)}$ are 0, $\frac{n-\mu_i(G)}{r}$ for $1 \leq i \leq n-1$.

Lemma 1. 2. (Jin and Ruibin (1999)). The $n-$dimensional hypercube Q_n has $n+1$ distinct eigenvalues. They are given by $\lambda_i(G) = -n + 2i$, and the eigenvalue $\lambda_i(G)$ has multiplicity $\binom{n}{i}$ for $0 \leq i \leq n$ where $n \geq 1$, $\binom{n}{i}$ is binomial coefficient.

Throughout this paper our notation is standard and taken mainly from the books (Bondy and Murty (1979); Brouwer and Haemers (2012); Chung (1997); Cvetković, Doob and Sachs (1980); Harary (1969)). The spectrum of an $n-$ vertex graph G is denoted by Spec(G) $=\begin{pmatrix} \lambda_1(G) & \lambda_2(G) & \cdots & \lambda_s(G) \\ t_1 & t_2 & \cdots & t_s \end{pmatrix}$, where s denotes the number of distinct eigenvalues of G and t_i is the multiplicity of $\lambda_i(G)$, $1 \leq i \leq s$.

2. SPECTRAL DISTANCES BETWEEN A HYPERCUBE AND ITS COMPLEMENT

The aim of this section is to compute the adjacency, Seidel, Laplacian signless Laplacian, normalized Laplacian spectral distances of Hypercube and its complement.

Proroposition 2. 1. If $n > 2$ then $\sigma(\overline{Q_n}, Q_n) = 2(2^n - n - 1)$.

Proof. By Lemma 1.2, if n is even then the eigenvalues of Q_n are as follows:

$$\begin{pmatrix} -n & -n+2 & \cdots & -2 & 0 & 2 & \cdots & n-2 & n \\ \binom{n}{0} & \binom{n}{1} & & \binom{n}{\frac{n}{2}-1} & \binom{n}{\frac{n}{2}} & \binom{n}{\frac{n}{2}+1} & & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix},$$

and if n is odd then the eigenvalues of Q_n are as follows:

$$\begin{pmatrix} -n & -n+2 & \cdots & -1 & 1 & \cdots & n-2 & n \\ \binom{n}{0} & \binom{n}{1} & & \binom{n}{\lfloor\frac{n}{2}\rfloor} & \binom{n}{\lceil\frac{n}{2}\rceil} & & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix}.$$

By Lemma 1.1(i), if n is even then the eigenvalues of $\overline{Q_n}$are as follows:

$$\begin{pmatrix} -n+1 & -n+3 & \cdots & -1 & 1 & \cdots & n-3 & n-1 & 2^n-n-1 \\ \binom{n}{1} & \binom{n}{2} & & \binom{n}{\frac{n}{2}-1} & \binom{n}{\frac{n}{2}} & & \binom{n}{n-1} & \binom{n}{n} & \binom{n}{0} \end{pmatrix},$$

and if n is odd then the eigenvalues of $\overline{Q_n}$ are as follows:

$$\begin{pmatrix} -n+1 & -n+3 & \cdots & 0 & 2 & \cdots & n-1 & 2^n-n-1 \\ \binom{n}{1} & \binom{n}{2} & & \binom{n}{\lfloor\frac{n}{2}\rfloor} & \binom{n}{\lceil\frac{n}{2}\rceil} & & \binom{n}{n} & \binom{n}{0} \end{pmatrix}.$$

Then by definition, we have

$$\begin{aligned} \sigma(\overline{Q_n}, Q_n) &= \textstyle\sum_{i=1}^{2^n} |\lambda_i(\overline{Q_n}) - \lambda_i(Q_n)| \\ &= (2^n - 2n - 1) + 1 + \left(\binom{n}{n-1} - 1\right) + \cdots + \left(\binom{n}{1} - 1\right) + 1 \\ &= 2^n - 2n + \textstyle\sum_{i=1}^{n-1} \binom{n}{i} = 2^{n+1} - 2n - 2, \end{aligned}$$

where the last equality follows from $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

■

Proposition 2. 2. If $n > 4$ then $S\sigma(\overline{Q_n}, Q_n) = 2^{n+2} - 8n$.

Proof. By Lemma 1.1 (ii) and Lemma 1.2, the Seidel eigenvalues of Q_n are as follows:

$$\begin{pmatrix} -2n+3 & -2n+7 & \cdots & 2n-5 & 2n-1 & 2^n-2n+1 \\ \binom{n}{1} & \binom{n}{2} & & \binom{n}{n-1} & \binom{n}{n} & \binom{n}{0} \end{pmatrix},$$

By Lemma 1.2 (iii), the Seidel eigenvalues of $\overline{Q_n}$ are as follows:

$$\begin{pmatrix} 1+2n-2^n & -2n+1 & -2n+5 & \cdots & 2n-7 & 2n-3 \\ \binom{n}{n} & \binom{n}{0} & \binom{n}{1} & & \binom{n}{n-2} & \binom{n}{n-1} \end{pmatrix}.$$

Then by definition, we have

$$\begin{aligned} S\,\sigma(\overline{Q_n}, Q_n) &= \sum_{i=1}^{2^n} |S\lambda_i(\overline{Q_n}) - S\lambda_i(Q_n)| \\ &= 2(2^n - 4n + 2) + 4 + 2\left(\binom{n}{n-1} - 2\right) + \cdots + 2\left(\binom{n}{1} - 2\right) \\ &\quad + 4(n-2) \\ &= 2^{n+1} - 4n + 2\sum_{i=1}^{n-1}\left(\binom{n}{i} - 2\right) = 2^{n+2} - 8n, \end{aligned}$$

where the last equality follows from $\sum_{i=0}^{n}\binom{n}{i} = 2^n$.

■

Proposition 2. 3. If $n > 2$ then

$$Q\sigma(\overline{Q_n}, Q_n) = L\sigma(\overline{Q_n}, Q_n) = nN\sigma(\overline{Q_n}, Q_n) = 2^n(2^n - 2n - 1).$$

Proof. By Lemma 1.1 (vi) and Lemma 1.2, if n is even then the Laplacian eigenvalues of Q_n are:

$$\begin{pmatrix} 0 & 2 & \cdots & n-2 & n & n+2 & \cdots & 2n-2 & 2n \\ \binom{n}{0} & \binom{n}{1} & & \binom{n}{\frac{n}{2}-1} & \binom{n}{\frac{n}{2}} & \binom{n}{\frac{n}{2}+1} & & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix},$$

and if n is odd then the eigenvalues of Q_n are as follows:

$$\begin{pmatrix} 0 & 2 & \cdots & n-1 & n+1 & \cdots & 2n-2 & 2n \\ \binom{n}{0} & \binom{n}{1} & & \binom{n}{\lfloor\frac{n}{2}\rfloor} & \binom{n}{\lceil\frac{n}{2}\rceil} & & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix}.$$

By Lemma 1.1 (v), if n is even then the Laplacian eigenvalues of $\overline{Q_n}$ are:

$$\begin{pmatrix} 0 & 2^n-2n & 2^n-2n+2 & \cdots & 2^n-2n & \cdots & 2^n-4 & 2^n-2 \\ \binom{n}{n} & \binom{n}{0} & \binom{n}{1} & & \binom{n}{\frac{n}{2}} & & \binom{n}{n-2} & \binom{n}{n-1} \end{pmatrix},$$

and if n is odd then the Laplacian eigenvalues of $\overline{Q_n}$ are as follows:

$$\begin{pmatrix} 0 & 2^n-2n & 2^n-2n+2 & \cdots & 2^n-2n-3 & 2^n-2n-1 & \cdots & 2^n-4 & 2^n-2 \\ \binom{n}{n} & \binom{n}{0} & \binom{n}{1} & & \binom{n}{\lfloor\frac{n}{2}\rfloor} & \binom{n}{\lceil\frac{n}{2}\rceil} & & \binom{n}{n-2} & \binom{n}{n-1} \end{pmatrix}.$$

Then by definition, we have

$$L\,\sigma(\overline{Q_n}, Q_n) = \sum_{i=1}^{2^n} |\mu_i(\overline{Q_n}) - \mu_i(Q_n)|$$

$$= (2^n - 2n - 2) + \left(\binom{n}{n-1} - 1\right)(2^n - 2n)$$
$$+(2^n - 2n - 2) + \left(\binom{n}{n-2} - 1\right)(2^n - 2n) + \cdots$$
$$+(2^n - 2n - 2) + \left(\binom{n}{1} - 1\right)(2^n - 2n) + (2^n - 2n - 2)$$
$$= n(2^n - 2n - 2) + (2^n - 2n)\sum_{i=1}^{n-1}\left(\binom{n}{i} - 1\right)$$
$$= n(2^n - 2n - 2) + (2^n - 2n)(2^n - n - 1).$$

Since from $\sum_{i=1}^{n-1}\left(\binom{n}{i} - 1\right) = 2^n - n - 1$, therefore $L\sigma(\overline{Q_n}, Q_n) = 2^n(2^n - 2n - 1)$. Since Q_n is bipartite, by (Brouwer and Haemers 2012, p. 17) Laplacian eigenvalues of Q_n are equal to the signless Laplacian eigenvalues of Q_n, by Lemma 1.1(xi) and Lemma 1.2, the signless Laplacian eigenvalues of $\overline{Q_n}$ are as follows:

$$\begin{pmatrix} 2^n-2n & 2^n-2n+2 & \cdots & 2^n-4 & 2^n-2 & 2(2^n-n-1) \\ \binom{n}{1} & \binom{n}{1} & & \binom{n}{n-1} & \binom{n}{n} & \binom{n}{0} \end{pmatrix}.$$

By definition, we get

$$Q\,\sigma(\overline{Q_n}, Q_n) = \sum_{i=1}^{2^n}|q_i(\overline{Q_n}) - q_i(Q_n)|$$
$$= (2^{n+1} - 4n - 2) + (2^n - 2n) + \left(\binom{n}{1} - 1\right)(2^n - 2n - 2)$$
$$+(2^n - 2n) + \cdots + \left(\binom{n}{n-1} - 1\right)(2^n - 2n - 2) + (2^n - 2n)$$
$$= 2(2^n - 2n - 1) + n(2^n - 2n)$$
$$+\sum_{i=1}^{n-1}\left(\binom{n}{i} - 1\right)(2^n - 2n - 2).$$

Therefore $Q\sigma(\overline{Q_n}, Q_n) = 2^n(2^n - 2n - 1)$. By Lemma 1.1(vii) and Lemma 1.2 we get the normalized Laplacian eigenvalues of Q_n as follows:

$$\begin{pmatrix} 0 & \frac{2}{n} & \frac{4}{n} & \cdots & \frac{2n-4}{n} & \frac{2n-2}{n} & 2 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & & \binom{n}{n-2} & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix}.$$

Now by Lemma 1.1(ix) the normalized Laplacian eigenvalues $\overline{Q_n}$ are as follows:

$$\begin{pmatrix} 0 & \frac{2^n-2n}{n} & \frac{2^n-2n+2}{n} & \cdots & \frac{2^n-4}{n} & \frac{2^n-2}{n} \\ \binom{n}{n} & \binom{n}{0} & \binom{n}{1} & & \binom{n}{n-2} & \binom{n}{n-1} \end{pmatrix},$$

and by definition,

$$N\sigma(\overline{Q_n}, Q_n) = \sum_{i=1}^{2^n}|\delta_i(\overline{Q_n}) - \delta_i(Q_n)|$$

$$= \frac{1}{n}((2^n - 2n - 2) + \left(\binom{n}{n-1} - 1\right)(2^n - 2n)$$
$$+(2^n - 2n - 2) + \left(\binom{n}{n-2} - 1\right)(2^n - 2n) + \cdots$$
$$+(2^n - 2n - 2) + \left(\binom{n}{1} - 1\right)(2^n - 2n) + (2^n - 2n - 2))$$
$$= (2^n - 2n - 2) + \frac{1}{n}(2^n - 2n)\sum_{i=1}^{n-1}\left(\binom{n}{i} - 1\right)$$
$$= (2^n - 2n - 2) + \frac{1}{n}(2^n - 2n)(2^n - n - 1),$$

where the last equality follows from $\sum_{i=0}^{n}\binom{n}{i} = 2^n$. This implies that $L\sigma(\overline{Q_n}, Q_n) = nN\sigma(\overline{Q_n}, Q_n)$. ■

3. SPECTRAL DISTANCES BETWEEN A LINE GRAPH HYPERCUBE AND ITS COMPLEMENT

The aim of this section is to compute the spectral distance of $l^k(Q_n)$ and its complement. For simplicity, we define $l^1(G) = l(G)$ and $l^k(G) = l(l^{k-1}(G))$, $k \geq 2$. It is both consistent and convenient to set $l^0(G) = l(G)$. If G is regular, then its line graph is also regular. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (r_0 n_0)/2$ and of degree $r_1 = 2r_0 - 2$. Therefore, the order and degree of $l^k(G)$ are $n_k = (r_{k-1}n_{k-1})/2$ and $r_k = 2r_{k-1} - 2$ where n_{k-1} and r_{k-1} are, respectively, the order and degree of $l^{k-1}(G)$. If the eigenvalues of G are $\lambda_i(G)$, $1 \leq i \leq n_0$ (arranged in increasing order), then the respective eigenvalues of $l(G)$ are -2 with multiplicity $n_1 - n_0$ and $\lambda_i(G) + r_0 - 2$ for $1 \leq i \leq n_0$) Cvetković at al. 1980). By Lemma 1.1(i) the eigenvalues of $\overline{l(G)}$ are as follows:

$$\begin{pmatrix} \left(\frac{n}{2}-2\right)r_0+1 & 1 & -1-(\lambda_1(G)+r_0-2) & \cdots & -1-(\lambda_{n-1}(G)+r_0-2) \\ 1 & \frac{n}{2}(r_0-2) & 1 & & 1 \end{pmatrix}.$$

Proposition 3. 1. If $n > 3$ then

$$\sigma\left(\overline{l(Q_n)}, l(Q_n)\right) = 2^n(4n - 9) - 10n + 24.$$

Proof. By above discussion, and Lemma 1.2 the eigenvalues of $l(Q_n)$ and $\overline{l(Q_n)}$ are as follows:

$$\begin{pmatrix} -2 & 0 & \cdots & 2n-6 & 2n-4 & 2n-2 \\ n2^{n-1}-2n+\binom{n}{0} & \binom{n}{1} & & \binom{n}{n-2} & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix},$$

$$\begin{pmatrix} -2n+3 & -2n+5 & \cdots & -3 & -1 & 1 & n2^{n-1}-2n+1 \\ \binom{n}{1} & \binom{n}{2} & & \binom{n}{n-2} & \binom{n}{n-1} & n2^{n-1}-2n+\binom{n}{n} & \binom{n}{0} \end{pmatrix}.$$

Then by definition, we have

$$\begin{aligned}\sigma\left(\overline{l(Q_n)}, l(Q_n)\right) &= \sum_{i=1}^{n2^{n-1}} \left|\lambda_i\left(\overline{l(Q_n)}\right) - \lambda_i(l(Q_n))\right| \\ &= n2^{n-1} - 4n + 3 + 2\left(\binom{n}{n-1}(2n-5) + \binom{n}{n-2}(2n-7)\right. \\ &\quad \left. + \cdots + 3\binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right) + 3(n2^{n-1} - 2^n - 2n + 3) \\ &= n2^{n+1} + (3 \times 2^n) - 8n + 12 + 2\sum_{i=2}^{n-1}\binom{n}{i}(2i-3) \\ &= 2^n\,(4n-9) - 10n + 24,\end{aligned}$$

where the last equality follows from $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ and from $\sum_{i=2}^{n-1}\binom{n}{i}i = n2^{n-1} - 2n$.

■

Analogously, the eigenvalues of the matrix $l^k(G)$ are:

$$\begin{pmatrix} -2 & \lambda_1\left(l^{k-1}(G)\right)-r_{k-1}-2 & \cdots & \lambda_n\left(l^{k-1}(G)\right)-r_{k-1}-2 \\ n_k-n_{k-1} & 1 & & 1 \end{pmatrix},$$

and the eigenvalues of the matrix of $l^k(G)$, $1 \le i \le n-1$, are:

$$\begin{pmatrix} n_k-r_{k-1}-1 & 1 & -1-\lambda_i\left(l^{k-1}(G)\right)+r_{k-1}+2 \\ 1 & n_k-n_{k-1} & 1 \end{pmatrix}$$

The spectral distance between $k-$iterated line graph, $k \ge 2, n > 3$ of a hypercube and its complement is:

$$\begin{aligned}\sigma\left(\overline{l^k(Q_n)}, l^k(Q_n)\right) &= n_k - 2r_k + r_0 - \lambda_n(Q_n) - 1 + 2(\sum_{i=1}^{n_0-1}(\lambda_i(Q_n) \\ &\quad + r_k - r_0 - 1)(n_1 - n_0 + 1)(r_k - r_1 - 3) \\ &\quad + \sum_{j=1}^{k-3}\left(n_{j+1} - n_j\right)\left(r_k - r_{j+1} - 3\right) \\ &\quad + (n_{k-1} - n_{k-2})(r_{k-1} - 5) \\ &\quad + 3(n_k - n_{k-1} - \left(\sum_{i=0}^{n-1}\binom{n}{i} + \sum_{j=0}^{k-1}\left(n_{j+1} - n_j\right)\right).\end{aligned}$$

From（Walikar, Ramane, Gutman and Halkarni (2007)), we have:

(a) $r_k = 2^k r_0 - 2^{k+1} + 2,$
(b) $n_k = \frac{n_0}{2^k}\prod_{j=0}^{k-1} r_j = \frac{n_0}{2^k}\prod_{j=0}^{k-1}(2^j r_0 - 2^{j+1} + 2).$

So that,

$$r_k - r_z = \sum_{i=z}^{k-1}(r_i - 2) = (r_0 - 2)(2^k - 2^z),$$

$$n_{j+1} - n_j = \frac{n_0}{2^j}(r_{j-1} - 2)\prod_{m=0}^{j-1} r_m.$$

Clearly, if G is an $n-$ vertex graph then $\sum_{i=1}^{n} \lambda_i(G) = 0$. So that $\sum_{i=1}^{n_0-1} \lambda_i(Q_n) = 0$. Since $n_0(Q_n) = 2^n$, $n_1(Q_n) = n2^{n-1}$, $r_0(Q_n) = n$, $\lambda_n(Q_n) = n$, that

$$\begin{aligned}\sigma\left(\overline{l^k(Q_n)}, l^k(Q_n)\right) = 2(&(2^n - 1)(r_k + n - 1) \\ &+(2^{n-1}(n-2)+1)((n-2)(2^k - 2) - 3) \\ &+\sum_{j=1}^{k-3}(n_{j+1} - n_j)\,((n-2)(2^k - 2^{j+1}) - 3) \\ &+ (n_{k-1} - n_{k-2})(r_{k-1} - 5)) + n_k - 2r_k - 1 \\ &+3\left(n_k - n_{k-1} - 2^n + 1 - \sum_{j=0}^{k-1}(n_{j+1} - n_j)\right).\end{aligned}$$

Here, $(n_{j+1} - n_j)(Q_n) = 2^{n-1}(n-2)\prod_{m=0}^{j-1} r_m$.

Corollary 3. 2. If $k = 2, 3$ and $n > 3$ then

$\sigma(\overline{l^2(Q_n)}, l^2(Q_n)) = n(2^{n+1}(2n-5) - 16) + 28,$

$\sigma(\overline{l^3(Q_n)}, l^3(Q_n)) = n(2^{n+1}(4n^2 - 13n + 9) - 32) + 60.$

4. LAPLACIAN SPECTRAL DISTANCES BETWEEN A LINE GRAPH HYPERCUBE AND ITS COMPLEMENT

The aim of this section is to compute the Laplacian spectral distance of $l^k(Q_n)$ and its complement. Let G be a regular graph with n vertex and m edge of degree r and $\mu_1(G), \dots, \mu_n(G)$ are the Laplacian eigenvalue of G. Since $\mu_i(G) = r - \lambda_i(G)$, $1 \le i \le n$, then the spectrum of the Laplacian of $l(G)$ is the same as the spectrum of the Laplacian of G, except that it has $(rn)/\,n$ extra eigenvalues of $2r$.

Proposition 4. 1. If $n > 3$ then

$$L\sigma(\overline{l(Q_n)}, l(Q_n)) = n2^{n-1}(n2^{n-1} - 4n + 3).$$

Proof. By above discussion, Lemma 1.1(iv) and Lemma 1.2, the Laplacian eigenvalues of $l(Q_n)$ are as follows:

$$\begin{pmatrix} 0 & 2 & \cdots & 2n-4 & 2n-2 & 2n \\ \binom{n}{0} & \binom{n}{1} & & \binom{n}{n-2} & \binom{n}{n-1} & n2^{n-1} - 2^n + \binom{n}{n} \end{pmatrix},$$

and the Laplacian eigenvalues of $\overline{l(Q_n)}$ are as follows:

$$\begin{pmatrix} 0 & n2^{n-1}-2n & n2^{n-1}-2n+2 & \cdots & n2^{n-1}-4 & n2^{n-1}-2 \\ \binom{n}{n} & n2^{n-1}-2^n+\binom{n}{0} & \binom{n}{1} & & \binom{n}{n-2} & \binom{n}{n-1} \end{pmatrix}.$$

Then by definition, we have

$$\begin{aligned} L\sigma\left(\overline{l(Q_n)}, l(Q_n)\right) &= \sum_{i=1}^{n2^{n-1}}\left|\mu_i\left(\overline{l(Q_n)}\right) - \mu_i(l(Q_n))\right| \\ &= 2(\tbinom{n}{1})(n2^{n-1} - 4n + 2) + \tbinom{n}{2}(n2^{n-1} - 4n + 4) \\ &\quad + \cdots + \tbinom{n}{n-1}(n2^{n-1} - 2n + 2)) + (n2^{n-1} - 2^{n+1} + 3\big) \\ &\quad (n2^{n-1} - 4n) \\ &= 2\sum_{i=1}^{n-1}\tbinom{n}{i}(n2^{n-1} - 2(n+i)) \\ &\quad + n2^{n-1}\big((n-4)2^{n-1} - 4n + 19\big) - 12n. \end{aligned}$$

Since $\sum_{i=1}^{n-1}\binom{n}{i} = 2^n - 2$ and $\sum_{i=1}^{n-1}\binom{n}{i}i = n2^{n-1} - n$,

$$\sum_{i=1}^{n-1}\binom{n}{i}\big(n2^{n-1} - 2(n+i)\big) = n2^{n-1}(2^n - 8) + 6n.$$

Therefore, $L\sigma(\overline{l(Q_n)}, l(Q_n)) = n2^{n-1}(n2^{n-1} - 4n + 3)$, which proves the result.

■

Analogously, the eigenvalues of the matrix L, $l^k(G)$ for $2 \le i \le n-1$ are:

$$\begin{pmatrix} 0 & \mu_i(l^{k-1}(G)) & 2r_0 & 2r_1 & \cdots & 2r_{k-1} \\ 1 & 1 & n_1-n_0+1 & n_2-n_1 & & n_k-n_{k-1} \end{pmatrix},$$

and the eigenvalues of the matrix L, $\overline{l^k(G)}$, $1 \le i \le n-1$, are:

$$\begin{pmatrix} 0 & n_k-2r_{k-1} & \cdots & n_k-2r_1 & n_k-2r_0 & n_k-\mu_i(l^{k-1}(G)) \\ 1 & n_k-n_{k-1} & & n_2-n_1 & n_1-n_0+1 & 1 \end{pmatrix}.$$

The Laplacian spectral distance between $k-$iterated line graph, $k \ge 2$ and $n > 3$ of a hypercube and its complement is:

$$\begin{aligned} L\sigma(\overline{l^k(Q_n)}, l^k(Q_n)) &= 2(\sum_{i=1}^{n_0-1} n_k - 2r_{k-1} - \mu_i(Q_n)) \\ &\quad + (n_1 - n_0 + 1)(n_k - 2(r_{k-1} - r_0)) \\ &\quad + \sum_{j=1}^{k-2}\big(n_{j+1} - n_j\big)\,(n_k - 2(r_{k-1} - r_j))) \\ &\quad + (n_k - n_{k-1} - (\sum_{i=0}^{n-1}\binom{n}{i} + \sum_{j=0}^{k-2}\big(n_{j+1} - n_j\big)) \\ &\quad (n_k - 4r_{k-1}). \end{aligned}$$

Since G is $n-$ vertex and m edges, $\sum_{i=1}^{n}\mu_i(G) = 2m$. So that, $\sum_{i=1}^{n_0-1}\mu_i(Q_n) = n(2^n - 2)$. By using (a) and (b), one can see that

$$L\sigma(\overline{l^k(Q_n)}, l^k(Q_n)) = 2(n2^{n-1}(n_k - 2(r_{k-1} + n + 1)) \\ + \sum_{j=1}^{k-2}(n_{j+1} - n_j)(n_k - 2(r_{k-1} + r_j))) \\ +(n_k - n_{k-1} - 2^n + 1 \\ - \sum_{j=0}^{k-2}(n_{j+1} - n_j))(n_k - 4r_{k-1}).$$

Here, $(n_{j+1} - n_j)(Q_n) = 2^{n-1}(n-2)\prod_{m=0}^{j-1} r_m$.

Corollary 4. 2. If $k = 2, 3$ and $n > 3$ then

$$L\sigma(\overline{l^2(Q_n)}, l^2(Q_n)) = n2^{n-1}(n2^{n-1}(n^2 - 2n + 1) - 8n^2 + 19n - 3) \\ -8n - 32(2^{n-1} - 1),$$

$$L\sigma(\overline{l^3(Q_n)}, l^3(Q_n)) = n2^{n-1}(n2^{n-1}(4n^4 - 20n^3 + 37n^2 - 30n + 9) \\ -32n^3 + 134n^2 - 183n + 81).$$

5. SIGNLESS LAPLACIAN SPECTRAL DISTANCES BETWEEN A LINE GRAPH HYPERCUBE AND ITS COMPLEMENT

The aim of this section is to compute the signless Laplacian spectral distance of $l^k(Q_n)$ and its complement. From (Cvetković and Simić (2009)). Let G be a regular graph with n vertex and m edge of degree r and $q_1(G), q_2(G), \ldots, q_n(G)$ be the signless Laplacian eigenvalues of G. Since $q_i(G) = \lambda_i(\mathrm{G}) + r,\ 1 \leq i \leq n$ the signless Laplacian eigenvalues of $l(G)$ are $q_1(G) + 2r - 4$, $q_1(G) + 2r - 4$, ..., $q_1(G) + 2r - 4$ and $2r - 4$ repeated $(r\,n)/2 - n$ times.

Proposition 5. 1. If $n > 3$ then

$$Q\sigma(\overline{l(Q_n)}, l(Q_n)) = n2^{n-1}(n2^{n-1} - 4n + 3).$$

Proof. By above discussion and Lemma 1.2 the signless Laplacian eigenvalues of $l(Q_n)$ are as follows:

$$\begin{pmatrix} 2(n-2) & 2n-2 & 2n & \cdots & 4n-6 & 4n-4 \\ n2^{n-1}-2^n+\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & & \binom{n}{n-1} & \binom{n}{n} \end{pmatrix}.$$

By Lemma 1.1(vii), the signless Laplacian eigenvalues of $\overline{l(Q_n)}$, are as follows:

$$\begin{pmatrix} n2^{n-1}-4n+4 & n2^{n-1}-4n+6 & \cdots & n2^{n-1}-2n & n2^{n-1}-2n+2 & Z_n \\ \binom{n}{1} & \binom{n}{2} & & \binom{n}{n-1} & n2^{n-1}-2^n+\binom{n}{n} & \binom{n}{0} \end{pmatrix}.$$

Here, $Z_n = 2(n2^{n-1} - 2n + 1)$. Then by definition,

$$\begin{aligned} Q\sigma\left(\overline{l(Q_n)}, l(Q_n)\right) &= \sum_{i=1}^{n2^{n-1}} \left|q_i\left(\overline{l(Q_n)}\right) - q_i(l(Q_n))\right| \\ &= (n2^n - 8n + 6) + 2\left(\binom{n}{n-1}(n2^{n-1} - 6n + 8)\right. \\ &\quad \left. + \cdots + \binom{n}{2}(n2^{n-1} - 4n + 2) + \binom{n}{1}(n2^{n-1} - 4n + 4)\right) \\ &\quad + (n2^{n-1} - 2^{n+1} + 3)(n2^{n-1} - 4n + 6) \\ &= n2^{n-1}(n2^{n-1} - 2^{n+1} - 4n + 27) - 20n + 24 \\ &\quad - 3 \times 2^{n+2} + 4\sum_{i=1}^{n-1}\binom{n}{i}(n2^{n-2} - 2n - i + 3). \end{aligned}$$

Since $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ and from $\sum_{i=0}^{n}\binom{n}{i}i = n2^{n-1}$,

$$\begin{aligned} \sum_{i=1}^{n-1}\binom{n}{i}(n2^{n-2} - 2n - i + 3) &= (2^{n-1} - 1)(n2^{n-1} - 5n + 6) \\ &= n2^{n-2} + 3 \times 2^n(1 - n) + 5n - 6. \end{aligned}$$

Therefore,

$$Q\sigma(\overline{l(Q_n)}, l(Q_n)) = n2^{n-1}(n2^{n-1} - 4n + 3).$$

This completes the proof.

■

Analogously, from (Gutman, Kiani, Mirzakhah and Zhou (2009)) the eigenvalues of the matrix Q, $l^k(G)$ are:

$$\begin{pmatrix} 2r_{k-1}-4 & q_1\left(l^{k-1}(G)\right)+2r_{k-1}-4 & \cdots & q_n\left(l^{k-1}(G)\right)+2r_{k-1}-4 \\ n_k-n_{k-1} & 1 & & 1 \end{pmatrix},$$

and the eigenvalues of the matrix Q, $\overline{l^k(G)}$, $1 \le i \le n-1$, are:

$$\begin{pmatrix} n_k-2r_{k-1}+2 & n_k-q_i\left(l^{k-1}(G)\right)-2r_{k-1}+2 & 2n_k-q_n\left(l^{k-1}(G)\right)-2r_{k-1}+2 \\ n_k-n_{k-1} & 1 & 1 \end{pmatrix}.$$

In what follows, the signless Laplacian spectral distances between k −iterated line graphs, $k \ge 2$ and $n > 3$, the line graph of hypercube and its complement are computed.

$$\begin{aligned} Q\sigma(\overline{l^k(Q_n)}, l^k(Q_n)) &= 2\left((n_k - n_0(Q_n) - 1 - 2\sum_{j=0}^{k-1}(r_j - 2)\right) \\ &\quad + \sum_{i=1}^{n_0-1}(n_k - q_i(Q_n) - 2(r_{k-1} + \sum_{j=1}^{k-1}(r_j - 2) - 1)) \\ &\quad +(n_1 - n_0 + 1)\left[n_k - 2(r_{k-1} + \sum_{i=0}^{k-1}(r_i - 2\right) - 1] \end{aligned}$$

$$+\sum_{j=1}^{k-2}(n_{j+1}-n_j)\,[n_k-2(r_{k-1}+\sum_{z=j}^{k-2}(r_z-2)-1])$$
$$+(n_k-n_{k-1}-(\sum_{i=0}^{n-1}\binom{n}{i}-\sum_{j=0}^{k-2}(n_{j+1}-n_j))$$
$$(n_k-4r_{k-1}+8).$$

By Euler's theorem, $\sum_{i=1}^{n} q_i(G)=2m$, where G have exactly n vertices and m edges. So, $\sum_{i=1}^{n_0-1} q_i(Q_n)= n(2^n-2)$. By using Equations (a) and (b), $q_0(Q_n) = 2n$ and,

$$\begin{aligned}Q\sigma(\overline{l^k(Q_n)}, l^k(Q_n)) = {} & 2(n_k-2^{k+1}(n-2)-n(2^n-2)-5\\ & +n\,2^{n-1}[n_k-2^{k+1}(n-2)-2(r_{k-1}-n+1)]\\ & +\sum_{j=1}^{k-2}(n_{j+1}-n_j)[n_k-2(r_{k-1}+(n-2)(2^k-2^j)\\ & -2(k-j)+1)]) + (n_k-n_{k-1}-\sum_{j=0}^{k-2}(n_{j+1}-n_j)\\ & -2^n+1)(n_k-4r_{k-1}+8),\end{aligned}$$

Here, $(n_{j+1}-n_j)(Q_n)=2^{n-1}(n-2)\prod_{m=0}^{j-1} r_m$.

Corollary 5. 2. If $k = 2, 3$ and $n>3$ then

$$Q\sigma(\overline{l^2(Q_n)}, l^2(Q_n)) = n2^{n-1}(n2^{n-1}(n^2-2n+1)-8n^2+21n-13)-2,$$

$$\begin{aligned}Q\sigma(\overline{l^3(Q_n)}, l^3(Q_n)) = {} & n2^{n-1}(n2^{n-1}(4n^4-20n^3+37n^2-30n+9)\\ & -32n^3+134n^2-183n+81).\end{aligned}$$

6. CONCLUDING REMARKS

In this paper some types of spectral distances between a hypercube and its complement, as well as the line graph of a hypercube and its complement are investigated. In this section some easy observation together with an open question are presented.

Let G be a graph on the vertex set $\{v_1, v_2, \ldots, v_p\}$. Define a bipartite graph G^* with set $V(G^*)=\{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_p\}$ in which v_i is adjacent to u_i for each $1\le i\le p$ and v_i is adjacent to u_j if v_i is adjacent to v_j in G. The graph G^* is known as the extended double cover graph of G. It is easy to see that G^* is regular of degree $r+1$ if only if G is regular of degree r (Chen (2004)).

Lemma 6. 1. (Chen (2004)) Let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ be the spectrum of G. Then the spectrum of G^* consists of $\pm(\lambda_1(G)+1)$, $\pm(\lambda_2(G)+1), \cdots, \pm(\lambda_n(G)+1)$

Proposition 6. 2. If G_1 and G_2 are n −vertex graphs, with eigenvalues $\lambda_i(G_j)$, $j = 1,\ 2$ and $1 \leq i \leq n$ such that $\lambda_1(G_j) \leq \lambda_2(G_j) \leq \cdots \leq \lambda_n(G_j)$, then $\sigma(G_1^*, G_1^*) = 2\sigma(G_1, G_2)$.

Corollary 6. 3.

(i) If $n > 2$ then $\sigma(\bar{Q}_n^*, Q_n^*) = 2\,\sigma(\bar{Q}_n, Q_n)$, $L\sigma(\bar{Q}_n^*, Q_n^*) = 2\,L\sigma(\bar{Q}_n, Q_n)$, $Q\sigma(\bar{Q}_n^*, Q_n^*) = 2\,Q\sigma(\bar{Q}_n, Q_n)$, $N\sigma(\bar{Q}_n^*, Q_n^*) = 2\,N\sigma(\bar{Q}_n, Q_n)$.

(ii) If $n > 4$ then $S\sigma(\bar{Q}_n^*, Q_n^*) = 2\,S\sigma(\bar{Q}_n, Q_n)$.

(iii) If $n > 3$ then

$$\sigma\left(\overline{l(Q_n^*)}, l(Q_n^*)\right) = 2\,\sigma(\overline{l(Q_n)}, l(Q_n)),$$
$$L\sigma\left(\overline{l(Q_n^*)}, l(Q_n^*)\right) = 2\,L\sigma(\overline{l(Q_n)}, l(Q_n)),$$
$$Q\sigma\left(\overline{l(Q_n^*)}, l(Q_n^*)\right) = 2\,Q\sigma(\overline{l(Q_n)}, l(Q_n)).$$

(iv) If $k \geq 2$ and $n > 3$ then

$$\sigma\left(\overline{l^k(Q_n^*)}, l^k(Q_n^*)\right) = 2\,\sigma(\overline{l^k(Q_n)}, l^k(Q_n)),$$
$$L\sigma\left(\overline{l^k(Q_n^*)}, l^k(Q_n^*)\right) = 2\,L\sigma(\overline{l^k(Q_n)}, l^k(Q_n)),$$
$$Q\sigma\left(\overline{l^k(Q_n^*)}, l^k(Q_n^*)\right) = 2\,Q\sigma(\overline{l^k(Q_n)}, l^k(Q_n)).$$

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