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A Pursuit Problem Described by Infinite System of Differential Equations with Coordinate-Wise Integral Constraints on Control Functions

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ABSTRACT

We consider a pursuit differential game of one pursuer and one evader described by infinite system of first order differential equations. The coordinate-wise integral constraints are imposed on control functions of players. By definition pursuit is said to be completed if the state of system equals zero at some time. A sufficient condition of completion of pursuit is obtained. Strategy for the pursuer is constructed and an explicit formula for the guaranteed pursuit time is given.

Keywords: Differential game, Hilbert space, infinite system, pursuit, evasion, strategy.

1. INTRODUCTION

A considerable amount of literature has been published on differential games where the control functions of players are subjected to integral constraints, see, for example, Azimov (1975), Azamov and Samatov (2000), Chikrii and Belousov (2009), Ibragimov et al (2014), Ibragimov and Salleh (2012), Ibragimov and Satimov (2012), Ibragimov et al (2011), Ibragimov (2002), Krasovskii (1968), Kuchkarov et al (2013), Nikolskii

(1969), Pshenichnii and Onopchuk (1968), Satimov and Tukhtasinov (2007), Satimov and Tukhtasinov (2005), Tukhtasinov (1995), Ushakov (1972).

As it is known (see, for example, Avdonin and Ivanov (1989), Butkovskiy (1975), Chernous'ko (1992)) that if we use the decomposition method for the systems with distributed parameters of evolution type, then we certainly arrive at an infinite system of differential equations. The main constraints on control parameters of players in finite dimensional differential games are geometric, integral, and mixed constraints (see, for example, Nikolskii (1969)). Also these constraints in some respect can be preserved for the control problems described by infinite systems of differential equations.

In the papers by Ibragimov et al (2014), Ibragimov (2002), Satimov and Tukhtasinov (2007), Satimov and Tukhtasinov (2005) various differential game problems for parabolic and hyperbolic equations were studied under different constraints on controls of players. Here, the decomposition method and some approaches from work of Avdonin and Ivanov (1989) were used to obtain various sufficient conditions under which the game problems are solvable.

The paper of Tukhtasinov and Ibragimov (2011) is devoted to the problems of keeping the trajectory of system within some limits, with integral constraints being subjected to controls. In all of these problems, the control parameters are in the right hand sides of equations in additive form. To solve the stated problems the control methods of infinite systems of differential equations were applied.

Therefore, there is an important link between the control problems described by some partial differential equations and infinite systems of differential equations. The latter is of independent interest. The papers of Ibragimov (2004), Ibragimov et al (2014), Ibragimov (2002), Satimov and Tukhtasinov (2007), Satimov and Tukhtasinov (2005), Tukhtasinov (1995) were devoted to differential game problems described by infinite system of differential equations. However, infinite systems require the existence and uniqueness theorem. Therefore, in the paper of Ibragimov (2004) such theorem was proved for the first order infinite system, and then a game problem was studied. In the present paper, we study a differential game problem described by infinite system of differential equations of first order. The control functions of players are subjected to coordinate-wise integral constraints. It should be noted that for such constraints the existence and uniqueness theorem in the paper of Ibragimov (2004), in general, doesn't work since the integral $\sum_{i=1}^{\infty}\int_0^T |u_i(t)|^2\, dt$ may diverge. We obtain sufficient conditions of completion of pursuit from any points of the state space.

2. STATEMENT OF THE PROBLEM

We study a two-person zero sum differential game described by the following infinite system of differential equations

$$\dot{z}_i + \lambda_i z_i = u_i - v_i,\ \ i = 1, 2, ... \tag{1}$$

where $z_i, u_i, v_i \in R^{n_i}$, n_i is a positive integer, λ_i are given positive numbers which express the elasticity of medium, $u = (u_1, u_2, ...)$ and $v = (v_1, v_2, ...)$ are control parameters of the pursuer and the evader respectively. These parameters vary depending on time, and therefore they become the functions of time. We denote them by $u(t), v(t),\ 0 \le t < \infty$. Let T be an arbitrary number.

Definition 1. A vector function $u(t) = (u_1(t), u_2(t), ...),\ 0 \le t \le T$, with measurable coordinates $u_i(t) \in \mathbb{R}^{n_i}$, is called admissible control of the i-*th* pursuer if it satisfies the following integral constraint

$$\int_0^T |u_i(s)|^2\, ds \le \rho_i^2, \tag{2}$$

where ρ_i is given nonnegative number which we call i-control resource of the pursuer.

Definition 2. A vector function $v(t) = (v_1(t), v_2(t), ...),\ 0 \le t \le T$, with measurable coordinates $v_i(t) \in \mathbb{R}^{n_i}$, is called admissible control of the evader if it satisfies the following integral constraint

$$\int_0^T |v_i(s)|^2\, ds \le \sigma_i^2, \tag{3}$$

where σ_i is given nonnegative number which we call i-control resource of the evader.

Pursuit starts from the initial positions

$$z_i(0) = z_{i0},\ i = 1, 2, ... \tag{4}$$

at time $t = 0$ where $z_{i0} \in R^{n_i}$, $i = 1, 2, ...$.

If we replace the parameters u_i, v_i in the equation (1) by some admissible controls $u_i(t)$, $v_i(t)$, $0 \le t \le T$, then it follows from the theory of differential equations that the initial value problem (1), (4) has a unique solution on the time interval $[0, T]$. The solution

$$z(t) = (z_1(t), z_2(t), ...),\ 0 \le t \le T,$$

of infinite system of differential equations (1) is considered in the space of functions $f(t) = (f_1(t), f_2(t), ...)$ with absolutely continuous coordinates $f_i(t)$ defined on the interval $0 \le t \le T$.

Definition 3. A function of the form

$$U(t, v) = \psi(t) + v = (\psi_1(t) + v_1,\ \psi_2(t) + v_2, ...), 0 \le t \le T,$$

where $U_i(t, v_i) = \psi_i(t) + v_i \in R^{n_i}$, $i = 1, 2, ...,$ is called strategy of the pursuer if for any admissible control of the evader $v = v(t), 0 \le t \le T$, the following inequalities

$$\int_0^T |U_i(t, v_i(t))|^2\, dt \le \rho_i^2,\quad i = 1, 2, ...,$$

hold.

Definition 4. We say that pursuit can be completed for the time $T > 0$ in the differential game (1)–(4) from the initial position $z_0 = (z_{10}, z_{20}, ...)$ if there exists a strategy of the pursuer $U(t, v)$ such that for any admissible control of the evader $v_i(t), 0 \le t \le T$, the solution $z(t), 0 \le t \le T$, of the initial value problem

$$\begin{aligned} &\dot{z}_i + \lambda_i z_i = U_i(t, v(t)) - v_i(t),\ 0 \le t \le T, \\ &z_i(0) = z_{i0},\ i = 1, 2, ..., \end{aligned} \tag{5}$$

equals zero at some time τ, $0 \le \tau \le T$, i.e. $z_i(\tau) = 0$ at some τ, $0 \le \tau \le T$. In the sequel, the time T is called guaranteed pursuit time.

Problem. Find a guaranteed pursuit time in the game (1) – (4), and construct the strategy for pursuer that enables to complete the game for this time.

3. MAIN RESULT

The following theorem presents a sufficient condition of completion of the game (1) – (4).

Theorem. Let $\rho_i > \sigma_i$, $i = 1, 2, \ldots$, and $\sup\limits_{i \in N} \dfrac{|z_{i0}|}{\rho_i - \sigma_i} < \infty$. Then

$$T' = \sup T_i, \; T_i = \frac{1}{2\lambda_i} \ln\left(1 + 2\lambda_i \left(\frac{|z_{i0}|}{\rho_i - \sigma_i}\right)^2\right), \tag{6}$$

is a guaranteed pursuit time in the game (1) – (4).

Proof. The hypothesis of the theorem $\sup\limits_{i \in N} \dfrac{|z_{i0}|}{\rho_i - \sigma_i} < \infty$ implies that T' is finite. Let $v_i(t), 0 \le t \le T'$, $i = 1, 2, \ldots$, be an arbitrary control of the evader. Construct the strategy of the pursuer as follows.

$$U_i(s,v) = \begin{cases} -\dfrac{z_{i0} e^{\lambda_i s}}{|z_{i0}| \varphi(\lambda_i, T_i)} (\rho_i - \sigma_i) + v_i, & 0 \le s \le T_i, \\ v_i, & T_i < s \le T', \end{cases} \tag{7}$$

where

$$\varphi(\lambda_i, t) = \sqrt{\int_0^t e^{2\lambda_i s} ds}, \; t \ge 0.$$

Verify admissibility of the strategy of the pursuer (7). Indeed,

$$\int_0^{T'} |U_i(s, v_i(s))|^2 ds = \int_0^{T_i} \left| -\frac{z_{i0} e^{\lambda_i s}}{|z_{i0}| \varphi(\lambda_i, T_i)} (\rho_i - \sigma_i) + v_i(s) \right|^2 ds + \int_{T_i}^{T'} |v_i(s)|^2 ds$$

$$= \frac{(\rho_i - \sigma_i)^2}{\varphi^2(\lambda_i, T_i)} \int_0^{T_i} e^{2\lambda_i s} ds - 2 \frac{1}{|z_{i0}|} \frac{\rho_i - \sigma_i}{\varphi(\lambda_i, T_i)} \int_0^{T_i} e^{\lambda_i s} (z_{i0}, v_i(s)) ds + \int_0^{T'} |v_i(s)|^2 ds. \tag{8}$$

Since $\int_0^{T'} |v_i(s)|^2 \le \sigma_i^2$, then

$$\int_0^{T'} |U_i(s,v_i(s))|^2 ds \le (\rho_i - \sigma_i)^2 + 2\frac{1}{|z_{i0}|}\cdot\frac{\rho_i-\sigma_i}{\varphi(\lambda_i,T_i)}\int_0^{T_i} e^{\lambda_i s}(z_{i0},v_i(s))ds + \sigma_i^2.$$

Using the Cauchy-Schwartz inequality, yields

$$\int_0^{T'} |U_i(s,v_i(s))|^2 ds \le (\rho_i - \sigma_i)^2 + 2(\rho_i - \sigma_i)\sigma_i + \sigma_i^2,$$

which implies that

$$\int_0^{T'} |U_i(s,v_i(s))|^2 ds \le \rho_i^2.$$

Thus, the strategy (7) is admissible.

Show that $z_i(T_i) = 0$. Let $v_i(s), 0 \le s \le T'$, be any admissible control of the evader. Replacing u_i in the equation

$$\dot{z}_i + \lambda_i z_i = u_i - v_i,\ 0 \le t \le T_i,$$
$$z_i(0) = z_{i0}.$$

by

$$u_i(s,v_i(s)) = -\frac{z_{i0}e^{\lambda_i s}}{|z_{i0}|\varphi(\lambda_i,T_i)}(\rho_i - \sigma_i) + v_i(s),$$

we obtain that

$$z_i(t) = e^{-\lambda_i t} z_{i0} + \int_0^t e^{-\lambda_i (t-s)}(u_i(s,v_i(s)) - v_i(s))ds$$
$$= e^{-\lambda_i t}\left[z_{i0} + \int_0^t e^{\lambda_i s}(u_i(s,v_i(s)) - v_i(s))ds \right]. \quad (9)$$
$$= e^{-\lambda_i t}\left[z_{i0} - \frac{z_{i0}}{|z_{i0}|\varphi(\lambda_i,T_i)}(\rho_i - \sigma_i)\int_0^t e^{2\lambda_i s} ds \right].$$

Hence

$$z_i(T_i) = e^{-\lambda_i T_i} \frac{z_{i0}}{|z_{i0}|}\left[|z_{i0}| - (\rho_i - \sigma_i)\frac{\varphi^2(\lambda_i, T_i)}{\varphi(\lambda_i, T_i)} \right]$$

$$= e^{-\lambda_i T_i} \frac{z_{i0}}{|z_{i0}|}\left[|z_{i0}| - (\rho_i - \sigma_i)\varphi(\lambda_i, T_i) \right]. \qquad (10)$$

Since

$$\varphi(\lambda_i, T_i) = \sqrt{\int_0^{T_i} e^{2\lambda_i s} ds} = \sqrt{\frac{e^{2\lambda_i T_i} - 1}{2\lambda_i}},$$

where T_i is defined by (6), we conclude that $\varphi(\lambda_i, T_i) = \frac{|z_{i0}|}{\rho_i - \sigma_i}$. Then the expression in brackets in (10) equals zero, and hence $z_i(T_i) = 0$.

According to (7), $u_i(s, v_i(s)) = v_i(s), s > T_i$, and therefore the equality $z_i(t) = 0$ keeps to hold on the interval $[T_i, T']$ as well. Thus, we conclude that $z_i(T') = 0,\ i = 1, 2, \ldots$. This completes the proof of the theorem.

4. CONCLUSION

In this paper, we have studied a differential game described by an infinite system of differential equations. In the past, such differential games were studied in Hilbert spaces (see, for example, Ibragimov (2002, 2004, 2014), Satimov and Tukhtasinov (2005, 2007). In the present research, the state $z(t)$ doesn't need to be in a Hilbert space, because the series $\sum_{i=1}^{\infty} \rho_i^2$ may be divergent. The control functions of the players are subjected to coordinate-wise integral constraints. We have obtained sufficient conditions of completion game. Moreover, we have constructed a strategy for the pursuer in explicit form.

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