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Projection Methods for Solving Urysohn Integral Equations with Multiwavelet Bases

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ABSTRACT

Urysohn integral equations appear in many applications, for example it occurs in solving problems arising in economics, engineering and physics. Equations of this type have been used to model many thermostatic devices. In this paper the Galerkin and the Petrov-Galerkin methods have been used to solve the nonlinear integral equation of the Urysohn type. Alpert (1993) constructed a class of wavele bases and applied it to approximate solutions of the Fredholm second kind integral equations by the Galerkin method. We use Alpert multiwavelet bases with orthonormal Legendre polynomials to approximate the solution of nonlinear integral equation of the Urysohn type. The numerical examples show the good accuracy of the method .

Keywords: Urysohn integral equation, Fredholm, Petrov-Galerkin method, Galerkin method, Multiwavelet bases.

1. INTRODUCTION

In this paper, we study the nonlinear Urysohn integral equation:

$$x(t) = \int_0^1 k\big(t, s, x(s)\big)ds + f(t), \quad 0 \le t \le 1 \tag{1}$$

for known functions k and f, an unknown solution x is to be approximated.

Equation (1) was discussed by Urysohn (1924). Equations of this type appear in many applications, for example it occurs in solving problems arising in economics, engineering and physics (Zabreiko *et al* (1975)). Also equations of this type have been used to model many thermostatic devices; see Glashoff and Sprekels (1981) and Glashoff and Sprekels (1981). Several authors have considered the problem of establishing the existence of solutions for (1) using different techniques. For example see (El-Sayed *et al.* 2003) and O'Regan (1998). In Atkinson and Rotra (1987), equation (1) is considered in a nonlinear operator equation

$$x = \mathcal{K}(x) \tag{2}$$

with $\mathcal{K}$ a completely continuous mapping of a domain in the Banach space X into X.

Let X_n, $n \geq 1$ denote a sequence of finite dimensional approximating subspaces and let P_n be a projection of X onto X_n. The projection method for solving $x = \mathcal{K}(x)$ consists of solving $x_n = P_n\mathcal{K}(x_n)$.

This method was analyzed by (Krasnoselskii *et al.* 1972) and the rate of convergence of $\{x_n\}$ to the exact solution were obtained. Detailed convergence results for both Galerkin and collocation projection methods for nonlinear case have been studied extensively in Atkinson and Rotra (1987).

2. MULTIWAVELET BASES

We provide below a brief review of Alpert's wavelets (see Alpert (1993)). The wavelet bases for $L^2[0,1]$ is comprised of dilates and translates of a set of functions $h_1, h_2, \dots, h_n$. In particular, for k a positive integer, and for $m = 0,1, \dots$, we define a space S_m^k of piecewise polynomial functions:

$$\begin{aligned} \mathrm{S}_{\mathrm{m}}^{\mathrm{k}} \quad = \quad & \{\text{f: there striction off to the interval} \quad (2^{\text{-m}}\text{n}, 2^{\text{-m}}(\text{n}+1)) \text{ is a} \\ & \text{polynomial of degree less than k, for} \\ & \text{n} = 0, \dots, 2^{\text{m}} - 1 \text{ and f vanish elsewhere}\}. \end{aligned} \tag{3}$$

Note that

$$\dim S_m^k = 2^m k,$$

and

$$S_0^k \subset S_1^k \subset \cdots \subset S_m^k \subset \cdots.$$

The orthogonal complement of S_m^k in S_{m+1}^k is denoted by R_m^k so that $\dim R_m^k = 2^m k$ and

$$S_m^k \oplus R_m^k = S_{m+1}^k, \qquad R_m^k \perp S_m^k$$

Also note that

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k \tag{4}$$

The set of functions $h_1, h_2, \dots, h_k$ mentioned above is taken as an orthonormal bases for R_0^k. Since R_0^k is orthogonal to S_0^k, the first k moments of $h_1, h_2, \dots, h_k$ vanish,

$$\int_0^1 h_j(x) x^i dx = 0, \qquad i = 0,1, \dots, k-1 \tag{5}$$

The wavelet bases of Alpert is constructed by defining orthogonal systems

$$h_{j,m}^n(s) = 2^{\frac{m}{2}} h_j(2^m s - n), \quad j = 1. \dots, k, \qquad m, n \in \mathbb{Z}. \tag{6}$$

We refer the reader to Alpert (1993) for detailed constructions of $h_1, h_2, \dots, h_k$. The function $h_{j,m}^n$ generated in (6) becomes an orthonormal bases for R_m^k.

$$R_m^k = \text{linear span}\,\{h_{j,m}^n; \quad j = 1, \dots, k, \quad n = 0, \dots, 2^m - 1\}.$$

If $\{u_1, u_2, \dots, u_k\}$ denote an orthonormal bases for R_0^k, then the orthonormal system

$$B_k = \{u_j, j = 1, \dots, k\} \cup \{h_{j,m}^n; \quad j = 1, \dots, k;\ m = 0, \dots; n = 0, \dots, 2^m - 1\},$$

becomes the multiwavelet bases of order k for $L^2[0,1]$. In practice we take an arbitrarily large but a fixed value for m and use

$$\tilde{B}_k = \{u_j, \quad j = 1, \dots, k\} \cup \{h_{j,m}^n, \quad j = 1, \dots, k,$$

$$n = 0,1, \dots, 2^m - 1\} = \{b_j\}_{j=1}^{k(2^m+1)} \tag{7}$$

for an orthonormal bases for S_m^k. For example the S_1^4 space has the basis elements as the following form:

$$b_1 = 1$$
$$b_2 = \sqrt{3}(2t-1)$$
$$b_3 = \sqrt{5}(6t^2-6t+1)$$
$$b_4 = \sqrt{7}(20t^3-30t^2+12t-1)$$
$$b_5 = \begin{cases} \sqrt{\frac{15}{17}}(3-56t+216t^2-224t^3), & 0 \le t \le \frac{1}{2}; \\ \sqrt{\frac{15}{17}}(-61+296t-456t^2+224t^3), & \frac{1}{2} < t \le 1. \end{cases}$$
$$b_6 = \begin{cases} \sqrt{\frac{1}{21}}(-11+270t-1320t^2+1680t^3), & 0 \le t \le \frac{1}{2}; \\ \sqrt{\frac{1}{21}}(-619+2670t-3720t^2+1680t^3), & \frac{1}{2} < t \le 1. \end{cases}$$
$$b_7 = \begin{cases} \sqrt{\frac{35}{68}}(2-60t+348t^2-512t^3), & 0 \le t \le \frac{1}{2}; \\ \sqrt{\frac{35}{68}}(-222+900t-1188t^2+512t^3), & \frac{1}{2} < t \le 1. \end{cases}$$
$$b_8 = \begin{cases} \sqrt{\frac{5}{84}}(-2+72t-492t^2+840t^3), & 0 \le t \le \frac{1}{2}; \\ \sqrt{\frac{5}{84}}(-418+1608t-2028t^2+840t^3), & \frac{1}{2} < t \le 1. \end{cases}$$

The approximating power of the wavelets is given as follows (see Alpert (1993)).

Lemma 1. 1*Let* Q_m^k *be the orthogonal projection of* $L^2[0,1]$ *onto* S_m^k. *If* $f \in C^k[0,1]$, *then*

$$\left\|Q_m^k f - f\right\| \le 2^{-mk} \frac{2}{4^k k!} \sup_{x\in[0,1]} |f^{(k)}(x)|.$$

2. THE GALERKIN METHOD

Consider the following Urysohn integral equation:

$$x(t) = \int_0^1 k\big(t, s, x(s)\big)ds + f(t), \quad 0 \le t \le 1 \tag{8}$$

where k and f are known functions and x is the unknown function to be approximated.

Here we use the Galerkin method with multiwavelet bases to give numerical solutions of Urysohn integral equation.

Let $X = L^2[0,1]$. Alpert (1993) proved that $L^2[0,1] = \overline{\bigcup_{m=0}^{\infty} S_m^k}$, so we choose

$$X_n = S_m^k, \qquad m, k \in \mathbb{Z}, \quad m \ge 0, \quad k \ge 2.$$

Now we define the orthogonal projection operator as the following form:

$$P_n : X \to X_n$$
$$P_n(x(t)) = x_n(t)$$

so

$$x_n(t) = \sum_{i=1}^{n} c_i b_i(t) \tag{9}$$

where $\{b_i(t)\}_{i=1}^{n}$ are orthonormal bases elements of S_m^k and $n = 2^m k$.

In order to solve equation (8) we write it in operator form:

$$x = \mathcal{K}(x) + f \tag{10}$$

where $\mathcal{K}$ is a completely continuous operator defined on X as follows:

$$\mathcal{K}(x) = \int_0^1 k(t, s, x(s))ds, \qquad 0 \le t \le 1, \quad x \in \mathbb{R}.$$

In order to obtain the approximate solution $x_n(t)$, it needs to satisfy equation (10):

$$\begin{aligned} x_n &= \mathcal{K}(x_n) + f \\ x_n &= \int_0^1 k(t, s, x_n(s))ds + f(t) \end{aligned} \tag{11}$$

The expansion of k in this bases is given by the formula

$$k\big(t,s,x_n(s)\big) = \sum_{i,j=1}^{n} k_{ij} b_i(s) b_j(t) \tag{12}$$

where the coefficients k_{ij} are given by the expansion

$$k_{ij} = \int_0^1 \int_0^1 k(t,s,x_n(s)) b_i(s) b_j(t) ds dt.$$

Using (12) and (9), integral equation (11) is thereby approximated by the equation:

$$\sum_{i=1}^{n} c_i b_i(t) - \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} b_i(s) b_j(t) ds = f(t), \qquad 0 \le t \le 1$$

multiplying both sides of the above equations with bases elements $\{b_j(t)\}$ and then integrating from 0 to 1 with respect to t and using the orthonormality of the bases $\{b_j(t)\}$, we have:

$$c_j - \sum_{i=1}^{n} k_{ij} \left(\int_0^1 b_i(s) ds \right) = f_j, \qquad j = 1, \dots, n \tag{13}$$

where

$$f_j = \int_0^1 f(t) b_j(t) dt, \qquad j = 1, \dots, n$$

Now we have n equations with n unknowns which can be solved easily.

4. THE PETROV-GALERKIN METHOD

We begin this section with a brief review of the Petrov-Galerkin method. We follow closely the paper by Chen and Xu (1998). Let X be a Banach space and X^* its dual space of continuous linear functionals. For each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$, and X_n and Y_n are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n.$$

We further assume the following approximation property.

(H) If $x \in X$ and $y \in Y$, then there are sequences $\{x_n\}$, $\{y_n\}$ with $x_n \in X_n$, $y_n \in Y_n$ for all n such that $x_n \to x$, and $y_n \to y$.

Define, for $x \in X$, an element $P_n x \in X_n$ called the generalized best approximation from X_n to x with respect to Y_n by the equation

$$\langle x - P_n x, y_n \rangle = 0, \quad \forall y_n \in Y_n. \tag{14}$$

It is proved by Chen and Xu (1998), that for each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^{\perp} = \{0\} \tag{15}$$

where $X_n^{\perp} = \{x^* \in X^*: \quad \langle x, x^* \rangle = 0 \quad \text{forall} \quad x \in X_n\}$. When condition (15) is satisfied, P_n defines a projection; $P_n^2 = P_n$. Throughout the remainder of this paper, condition (15) is assumed.

In order to formulate the Petrov-Galerkin method as part of the general projection scheme and render an appropriate error analysis accordingly, it is important to establish that P_n converges pointwise to the identity operator I. To this end, the notion of regular pair is introduced. Assume that for each n, there is a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ and satisfying the following two conditions.

$$(H-1) \quad \|x_n\| \le C_1 \langle x_n, \Pi_n x_n \rangle^{\frac{1}{2}} \quad \text{forall} \quad x_n \in X_n.$$
$$(H-2) \|\Pi_n x_n\| \le C_2 \|x_n\| \quad \text{forall} \quad x_n \in X_n.$$

Here C_1 and C_2 are constants independent of n.

If a pair of space sequences $\{X_n\}$ and $\{Y_n\}$ satisfy (H-1) and (H-2), we call $\{X_n, Y_n\}$ a regular pair. It is shown by Chen and Xu (1998), that, if a regular pair $\{X_n, Y_n\}$ satisfies $\dim X_n = \dim Y_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

$$(P1) \quad \|P_n x - x\| \to 0 \quad \text{as} \quad n \to 0 \quad \text{for all} \quad x \in X,$$
$$(P2) \quad \|P_n\| \le C, \quad n = 1,2,\ldots, \quad \text{for some constant} \quad C, \quad \text{and}$$
$$(P3) \quad \|P_n x - x\| \le c \|Q_n x - x\|, \quad n = 1,2,\ldots, \quad \text{for some constant} \quad c,$$

where $Q_n x$ is the best approximation from X_n to x.

5. ESTABLISHMENT OF THE PETROV- GALERKIN CONDITIONS

In this section we show that all necessary conditions shown in previous section hold with Alpert Multiwavelet basis. At first we should choose the appropriate spaces for X and X^*. Let put $X = L^2[0,1]$, so the dual space will be the same: $X^* = L^2[0,1]$. (The dual of L^p space is L^q where $\frac{1}{p} + \frac{1}{q} = 1$).

We saw in section 2 that $X = L^2[0,1] = \overline{\bigcup_{m=0}^{\infty} S_m^k}$, which S_m^k, $m = 0,1,...$ are the spaces constructed from Alpert Multiwavelets. Now we introduce the space sequences X_n and Y_n:

$$X_n \quad = \quad S_m^k \subseteq \overline{\bigcup_{m=0}^{\infty} S_m^k} = L^2[0,1] = X$$

$$Y_n \quad = \quad S_{m\prime}^k{}' \subseteq \overline{\bigcup_{m\prime=0}^{\infty} S_{m\prime}^{k\prime}} = L^2[0,1] = X^*$$

In order to satisfy in condition $dimX_n = dimY_n < \infty$ we should have $2^m k = 2^{m\prime} k' < \infty$ and this equation implies that $2^{m-m\prime} = \frac{k}{k\prime}$.
In this article, we consider $m' = m + 1$ and $k' = \frac{k}{2}$.

Condition (H):

$$\forall x_n(t) \in X = L^2[0,1] = \overline{\bigcup_{m=0}^{\infty} S_m^k} \qquad \exists x_n(t) \in X_n = S_m^k; \quad \text{P}x_n(t) - x_t\text{P}_2 \to 0, \quad asn \to \infty.$$

$$\forall y_n(t) \in X^* = L^2[0,1] = \overline{\bigcup_{m'=0}^{\infty} S_{m'}^{k'}} \qquad \exists x_n(t) \in Y_n = S_{m'}^{k'}; \quad \text{P}y_n(t) - y_t\text{P}_2 \to 0, \quad asn \to \infty.$$

In above equations, $\|.\|_2$ is defined in the space $L^2[0,1]$. We introduce $x_n(t)$ and $y_n(t)$ as follows. If $\{b_i(t)\}_{i=1}^n$ be a basis for the space $X_n = S_m^k$ and $\{b_i^*(t)\}_{i=1}^n$ be another one for the space $Y_n = S_{m\prime}^{k\prime}$ with the condition $n = 2^m k = 2^{m\prime} k'$, then we define the projection operator P_n as the same as we did in Galerkin method:

$$P_n : X = \overline{\bigcup_{m=0}^{\infty} S_m^k} \to X_n = S_m^k$$
$$P_n(x(t)) = x_n(t) = \sum_{i=1}^{n} c_i b_i(t).$$

Also we can define the operator $P_n{}'$ from X^* to Y_n in a same way. Now we should define the operator $\prod_n$ which used in definition of regular pair:

$$\prod_n : X_n = S_m^k \to Y_n = S_{m'}^{k'}$$
$$\prod_n \left(x_n(t)\right) = \prod_n \left(\sum_{i=1}^{n} \alpha_i b_i(t)\right) = \sum_{i=1}^{n} \alpha b_i^*(t).$$

Whit this definition $\prod_n$ is clearly linear and one to one, so with respect to that $dimX_n = dimY_n$ we can conclude that $\prod_n X_n = Y_n$.

The issue we are concerned with is to show that conditions (H-1) and (H-2) are established.

(H-1): $\forall x_n(t) \in X_n = S_m^k \|x_n\| < c_1 \langle x_n, \prod_n x_n \rangle^{\frac{1}{2}}$.

Proof.

$$\begin{aligned}
\langle x_n, \prod_n x_n \rangle &= \int_0^1 x_n(t) \left(\prod_n x_n(t)\right) dt \\
&= \sum_{n=0}^{2^m-1} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} \left[\sum_{i=1}^{2^m k} \alpha_i b_i(t)\right] \left[\sum_{j=1}^{2^m k} \alpha_j b_j^*(t)\right] dt \\
&= \sum_{n=0}^{2^m-1} \sum_{i,j=1}^{2^m k} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} b_i(t) b_j^*(t) \alpha_i \alpha_j dt \\
&= \sum_{i,j=1}^{2^m k} \alpha_i \alpha_j \int_0^1 b_i(t) b_j^*(t) dt
\end{aligned}$$

Let $\alpha^T = (\alpha_1, \cdots, \alpha_n)$ and $B = [b_{i,j}]_{i,j=1}^{n} = \left[\int_0^1 b_i(t) b_j^*(t) dt\right]_{i,j=1} 2^m k$, then:

$$\langle x_n, \prod_n x_n \rangle = \alpha^T B \alpha \tag{16}$$

Now we choose spaces X_n and Y_n in such a way that the condition (H-1) holds. If k be a positive integer, by putting $X_n = S_0^k$ and $Y_n = S_1^{\frac{k}{2}}$, the matrix B will be diagonal with positive members. So we have:

$$\langle x_n, \prod_n x_n \rangle = \sum_{i=1}^{n} \alpha_i^2 b_{ii} \geq b_{qq} = b_{qq} \|x_n\|_2^2$$

Where $b_{qq} = \min_{1 \leq i \leq n} b_{ii}$, so

$$\langle x_n, \prod_n x_n \rangle \geq b_{qq} \|x_n\|$$

and

$$\|x_n\| \leq \frac{1}{\sqrt{b_{qq}}} \langle x_n, \prod_n x_n \rangle^{\frac{1}{2}}$$

Considering $c_1 = \frac{1}{\sqrt{b_{qq}}}$, condition (H-1) holds.

(H-2): $\forall x_n(t) \in X_n = S_m^k, \quad \|\prod_n x_n\| \leq c_2 \|x_n\|$

Proof.

$$\left\|\prod_n x_n\right\|_2^2 = \int_0^1 \left(\prod_n x_n(t)\right)^2 dt = \int_0^1 \left(\sum_{i=1}^{2^{m\prime}k\prime} \alpha_i b_i^*(t)\right)^2 dt$$

By dividing the interval [0,1] into subintervals $[\frac{n}{2^m}, \frac{n+1}{2^m}]$, $\quad n = 0,1, \dots, 2^m - 1$, we can continue the previous equation like this:

$$\|\Pi_n\, x_n\|_2^2 = \quad \sum_{n=0}^{2^m-1} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} \left(\sum_{i=1}^{2^m k} \alpha_i b_i^*(t)\right)^2 dt$$

$$= \quad \sum_{n=0}^{2^m-1} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} \left[\sum_{i=1}^{2^m k} [\alpha_i^2 b_i^{*2}(t) + 2\prod_{j=1,j\neq i}^{2^m k} \alpha_i \alpha_j b_i^*(t) b_j^*(t)]\right] dt$$

$$= \quad \sum_{i=1}^{2^m k} \alpha_i^2 \sum_{n=0}^{2^m-1} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} b_i^{*2}(t)dt + \sum_{i=1}^{2^m k} \left(\sum_{n=0}^{2^m-1} \int_{\frac{n}{2^m}}^{\frac{n+1}{2^m}} [2\prod_{j=1,j\neq i}^{2^m k} \alpha_i \alpha_j b_i^*(t) b_j^*(t)] dt\right)$$

$$= \quad \sum_{i=1}^{2^m k} \alpha_i^2 \int_0^1 b_i^{*2}(t)dt + \sum_{i=1}^{2^m k} \left(2\prod_{j=1,j\neq i}^{2^m k} \alpha_i \alpha_j\right) \int_0^1 b_i^*(t) b_j^*(t) dt$$

Orthonormality of Y_n basis elements implies that the second integral vanishes, so:

$$\left\|\prod_n x_n\right\|_2^2 = \sum_{i=1}^{2^m k} \alpha_i^2 \int_0^1 b_i^{*2}(t)dt$$

$$\Rightarrow \left\|\prod_n x_n\right\| = \|x_n\| \leq c_2\|x_n\|$$

Now it is sufficient to take $c_2 \geq 1$.

6. APPLYING THE METHOD

The Petrov-Galerkin approximation to the Urysohn integral equation (2) is obtained by solving the following equation for x_n

$$\langle x_n - \mathcal{K}x_n, b_j^* \rangle = \langle f, b_j^* \rangle$$

where $\{b_j^*(t)\}$ are the bases elements of subspace Y_n. Using (9) and the fact that $\mathcal{K}(x_n) \in X_n$, it follows that

$$\sum_{i=1}^{n} c_i b_{ij} - \sum_{i=1}^{n} \hat{c}_i b_{ij} = f_j \tag{16}$$

where $b_{ij} = \langle b_i(t), b_j^*(t) \rangle$ and $f_j = \langle f(t), b_j^*(t) \rangle$. Note that since $\mathcal{K}x_n \in X_n$, so

$$\mathcal{K}x_n = \sum_{i=1}^{n} \hat{c}_i b_i(t),$$

To modify the coefficients $\hat{c}_j$, by multiplying both sides of the above equation by $\{b_j(t)\}$ and integrate over the interval $[0,1]$ with respect to t, we have

$$\begin{aligned}\hat{c}_j &= \langle \mathcal{K}x_n(t), b_j(t)\rangle \\ &= \int_0^1\int_0^1 k\left(t,s,\sum_{i=1}^{n} c_i b_i(s)\right) b_j(t)dsdt.\end{aligned}$$

By substituting in Eq. (16)

$$\sum_{i=1}^{n} c_i b_{ij} - \sum_{i=1}^{n} b_{ij}\left[\int_0^1\int_0^1 k(t,s,\sum_{i=1}^{n} c_i b_i(s)) b_i(t)dsdt\right] = f_j, \quad j = 1,2,\dots,n$$

Let

$$K_i = \int_0^1\int_0^1 k\left(t,s,\sum_{i=1}^{n} c_i b_i(s)\right) b_i(t)dsdt,$$

so

$$\sum_{i=1}^{n} c_i b_{ij} = f_j + \sum_{i=1}^{n} K_i b_{ij}, \quad j = 1,\dots,n.$$

The iterated projection method solution is defined by

$$\tilde{x}_n = \mathcal{K}(x_n).$$

7. NUMERICAL EXAMPLES

In this section we illustrate the method discussed in section 3 with some examples.

Example 1. *In this example we solve the following equation*

$$x(t) = \int_0^1 \left(e^{t+s} + \frac{1}{2}x(s)\right)ds + 2e^t + \frac{1}{2} - \frac{1}{2}e - e^{(t+1)}$$

with the exact solution e^t. Numerical results are presented in Table 1.

Example 2. *In this example we solve the following equation*

$$x(t) = \int_0^1 \sin(t + s + x(s))ds + (t - \sin 1 \sin(1 + t))$$

with the exact solution t. Numerical results are presented in Table 2.

TABLE 1: Absolute errors for example 1(Galerkin method).

iteration	S_0^4	S_0^8	S_1^4	S_2^4
10	7.757 E--4	7.014 E--4	5.512 E--5	4.231 E--6
20	3.313 E--4	6.853 E--7	2.316 E--7	3.251 E--8
30	3.312 E--4	0.854 E--8	1.015 E--9	0.168 E--10
40	3.312 E--4	1.853 E--8	0.158 E--10	3.012 E--11

TABLE 2: Absolute errors for example 1(Petrov-Galerkin method).

iteration	$X_n = S_0^4$	$X_n = S_0^6$	$X_n = S_0^8$	$X_n = S_0^{10}$
	$Y_n = S_1^2$	$Y_n = S_1^3$	$Y_n = S_1^4$	$Y_n = S_1^5$
5	8.810 E--3	8.844 E--3	8.803 E--3	7.213 E--3
10	4.339 E--4	8.890 E--4	2.755 E--4	5.621 E--3
15	3.357 E--4	8.454 E--4	1.827E--4	5.121 E--4
20	3.355 E--4	8.454 E--4	1.612 E--5	2.257 E--5

TABLE 3: Absolute errors for example 2(Galerkin method).

iteration	S_0^4	S_0^8	S_1^4	S_2^4
10	6.521 E--4	6.254 E--4	3.612 E--4	2.158 E--5
20	3.359 E--4	5.268 E--6	2.158 E--7	6.268 E--7
30	3.759 E--5	4.268 E--8	3.124 E--9	4.728 E--10
40	2.168 E--5	3.684 E--9	1.589 E--10	2.154 E--12

CONCLUSION

The Galerkin and the Petrov-Galerkin methods have been used to solve the nonlinear integral equation of the Urysohn type by using a class of Alpert multiwavelets with orthonormal Legendre polynomials. The numerical examples show that the presented methods are effective and trustable.

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REFERENCES

Alpert, B.K. (1993). A class of bases in L^2 for the sparse representationof integral operators. *SIAM J. Math. Anal*. **24**: 246-262.

Atkinson, K.E., Rotra,F. A. (1987). Projection and iterated projection methods for nonlinear integral equations. *SIAM J. Math. Anal*. **24**: 1352-1373.

Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* in: Cambridge Monographs on Applied and Computetional Mathematics, Cambridge: Cambridge University Press, ,.

Chen, Z., Xu, Y. (1998). The Petrov-Galerkin and iterated Petrov-Galerkin methods for second kind integral equations. *SIAM J. Numer. Anal.* **35**: 406 - 434.

El-Sayed, W. G., El-Bary, A. A. and Darwish, M. A. (2003). Solvability of Urysohn integral equation. *Applied Mathematics and Computation*. **145**: 487-493.

Glashoff, K. and Sprekels, J. (1982). The Regulation of Temperature by Thermostats and Set-Valued Integral Equations. *Journal of Integral Equations*. **4**: 95-112.

Glashoff, K. and Sprekels, J. (1981). An Application of Gliksbergâ€™s Theorem to Set-Valued Integral Equations Arising in the Theory of Thermostats. *SIAM Journal on Mathematical analysis*. **12**: 477-486.

Jafarian, A., Esmailzadeh, Z., Khoshbakhti, L. (2013). A Numerical Method for SolvingNonlinear Integral Equations in the Urysohn Form. *Applied Mathematics Siences*. **7** (28): 1375-1385.

Kaneko, H., Noren, R. D. and Novaprateep, B. Wavelet application to the Petrov-Galerkin method for Hammerstein equations. *Applied Numerical Mathematics*. **45**: 255-273.

O'Regan, D. (1998). Volterra Urysohn integral equations in Banach spaces. *J. Appl. Math. Stochastic Anal.* **11**(4): 449-464.

Saberi-Nadjafi, J. and Heidari, M. (2010). Solving nonlinear integral equations in the Urysohn form by Newton-Kantorovich-quadrature method. *Computers and Mathematics with Applications*. **60**: 2058-2065.

Urysohn, P. S. (1924). On a type of nonlinear integral equation. *Mat. Sb*. **31**: 236-355.

Zabreiko, P. P., Koshelev, A. I, *et al*. (1975). *Integral equations: A Reference Text*. Leyden: Noordhoff Int. Publicaion.