Malaysian Journal of Mathematical Sciences 9(2): 325-336 (2015)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

The Interval Zoro-Symmetric Single-Step IZSS1-5D for the Simultaneous Bounding of Real Polynomial Zeros

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ABSTRACT

A new modified method IZSS1-5D for the simultaneously bounding all the real zeros of a polynomial is formulated in this paper. The efficiency of this method is measured on the CPU times and the number of iterations after satisfying the convergence criteria where the results are obtained using five tested polynomials. The analysis performed shows that the *R*-order of convergence of this new procedure is at least five. The programming language used to obtain the numerical results is Matlab R2012, a software in association with Intlab V5.5 toolbox. The numerical results indicate that the procedure IZSS1-5D outperformed the IZSS1 in computational times and number of iterations.

Keywords: CPU time, initial disjoint intervals, interval, number of iteration.

1. INTRODUCTION

Consider $p: R^1 \rightarrow R^1$ a polynomial of degree $n > 1$ defined by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = \prod_{j=1}^{n}(x - x_j^*) \qquad (1)$$

where $a_i \in R^1 (i=1,...,n)$ are given. Suppose that p has n distinct values $x_i^* \in R(i=1,...,n)$ and that $X_i^{(0)} \in I(R)$ (set of real intervals) $(i=1,...,n)$ are such that

$$x_i^* \in X_i^{(0)}, \ (i=1,...,n) \tag{2}$$

and

$$X_i^{(0)} \cap X_j^{(0)} = \varnothing, \quad (i,j=1,...,n; i \neq j). \tag{3}$$

There are researchers who came out with the new idea of applying the interval arithmetic, such as Moore (1979) and Alefeld and Herzberger (1983) in determining the convergence rate of the interval procedures related to bounding polynomial zeros. These were then followed by Bakar *et al.* (2012), Sham *et al.* (2013a, 2013b), Jamaludin *et al.* (2013a, 2013b, 2013c, 2013d, 2014a, 2014b) and Rusli *et al.* (2014a, 2014b). Interval iterative procedure for simultaneous inclusion of simple polynomial zeros determine bounded closed intervals which contain an exact polynomial zeros. It can be used to determine very narrow computationally rigorous bound on polynomial zeros. It is a very significant way of obtaining reliable bounds on the zeros as the intervals sequences generated by the procedures always converges to the zeros.

The aim of this paper is to present the interval zoro symmetric single-step procedure, IZSS1-5D which is the modification of interval zoro symmetric single-step procedure, IZSS1 (Rusli *et al.* 2011). IZSS1-5D method has been modified in order to increase the efficiency of the method. We repeat the same inner looping and add up another step to test the accuracy. The efficiency of the algorithm is measured numerically by taking the CPU time and also the number of iterations of the algorithm.

2. THE INTERVAL ZORO SYMMETRIC SINGLE-STEP PROCEDURE IZSS1-5D

An extension of the interval single-step procedure IS is the interval symmetric single-step procedure ISS. This idea is basically ideas of Alefeld and Herzberger (1983), and Monsi and Wolfe (1988). The procedure IZSS1-5D consists of generating the interval sequence $\{X_i^{(k)}\}, (i=1,...,n)$ from the following algorithm,

Step 1: Set $k=0$, (4a)

Step 2: For $k \geq 0, x_i^{(k)} = \text{mid}\left(X_i^{(k)}\right), \ (i=1,...,n);$ (4b)

Step 3: Let $\delta_j = \delta_{ij}^{(k)} = -p\left(x_i^{(k)}\right) \Big/ \prod_{j \neq i}^{n} (x_i^{(k)} - x_j^{(k)})$ (4c)

Step 4:

$$X_i^{(k,1)} = \left\{ x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\left\{ \prod_{j=1}^{i-1}\left(x_i^{(k)} - X_j^{(k,1)}\right) \prod_{j=i+1}^{n}\left(x_i^{(k)} - X_j^{(k,0)} - 5\delta_j\right) \right\}} \right\} \cap X_i^{(k)} \tag{4d}$$

$$(i = 1, \ldots, n)$$

Step 5:

$$X_i^{(k,2)} = \left\{ x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\left\{ \prod_{j=1}^{i-1}\left(x_i^{(k)} - X_j^{(k,1)}\right) \prod_{j=i+1}^{n}\left(x_i^{(k)} - X_j^{(k,2)}\right) \right\}} \right\} \cap X_i^{(k,1)} \tag{4e}$$

$$(i = n, \ldots, 1)$$

Step 6:

$$X_i^{(k,3)} = \left\{ x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\left\{ \prod_{j=1}^{i-1}\left(x_i^{(k)} - X_j^{(k,3)}\right) \prod_{j=i+1}^{n}\left(x_i^{(k)} - X_j^{(k,2)}\right) \right\}} \right\} \cap X_i^{(k,2)} \tag{4f}$$

$$(i = 1, \ldots, n)$$

Step 7: $X_i^{(k+1)} = X_i^{(k+1,0)} = X_i^{(k,3)}$ (4g)

Step 8: If the widths of intervals $X_i^{(k+1)} < \varepsilon$ $(i = 1, \ldots, n)$ then stop, else set $k = k + 1$ and go to Step 2.

The procedure IZSS1-5D has the following attractive features:

- The use of $5\delta_j \ (j=1,...,n)$ in (4d) will improve the efficiency of this procedure.
- The values $p(x_i^{(k)})(i=1,...,n)$ which are computed for use in (4d) are re-used in (4e) and (4f).
- The products $\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \ (i=2,...,n)$ which are computed for use in (4d) are re-used in (4e).
- The products $\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,2)}) \ (i=2,...,n)$ which are computed for use in (4e) are re-used in (4f).

The above procedure without $5\delta_j (j=1,...,n)$ in (4d) is the procedure IZSS1 in Rusli *et al.* (2011). It is an extension of the procedure ISS1-5D (Sham *et al.* 2013a).

3. THE RATE OF CONVERGENCE OF IZSS1-5D

The following theorem contains the conditions so that the algorithm in (4a)-(4g) converges to the zeros of the polynomial in (1).

Theorem 3.1: If (i) (2) and (3) hold; (ii) the sequences $\{X_i^{(k)}\}(i=1,...,n)$ are generated from (4), then $(\forall k \geq 0)\, x_i^* \in X_i^{(k+1)} \subseteq X_i^{(k)} \ (i=1,...,n)$. If also (iii) $0 \notin D_i \in I(R)$ is such that $p'(x) \in D_i \ (\forall x \in X_i^{(0)}) \ (i=1,...,n)$, then $X_i^{(k)} \to x_i^* \ (k \to \infty)$ and $(\forall k \geq 0) \ (i=1,...,n)$

$$w(X_i^{(k+1)}) \leq \frac{1}{2}\left(1 - \frac{d_{iI}}{d_{iS}}\right) w(X_i^{(k)}), \tag{5}$$

where $w(X_i^{(k)}) = w([x_{iI}^{(k)}, x_{iS}^{(k)}]) = x_{iS}^{(k)} - x_{iI}^{(k)}$. Hence, $O_R(IZSS1-5D, x_i^*) \geq 5$.

Proof.

The proof that $x_i^* \in X_i^{(k+1)} \subseteq X_i^{(k)} (i=1,...,n)(\forall k \geq 0)$ holds is almost identical with the corresponding proofs in Monsi and Wolfe (1988) and Ortega and Rheinboldt (1970), and is therefore omitted.

As the proof in Monsi *et al.* (2012), it may be shown that $\exists \alpha > 0$ such that $(\forall k \geq 0)$,

$$w_i^{(k,1)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} (w_j^{(k,0)})^2 \right\} \quad (i = 1,\dots,n), \tag{6}$$

$$w_i^{(k,2)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,2)} \right\} \quad (i = n,\dots,1), \tag{7}$$

and

$$w_i^{(k,3)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,3)} + \sum_{j=i+1}^{n} w_j^{(k,2)} \right\} \quad (i = 1,\dots,n), \tag{8}$$

where

$$w_i^{(k,s)} = (n-1)\ \alpha w\ (X_i^{(k,s)}) \quad (s = 0,1,2), \tag{9}$$

and

$$\beta = \frac{1}{n-1} \quad . \tag{10}$$

Let

$$u_i^{(1,1)} = \begin{cases} 3 & (i = 1,\dots,n\text{-}1) \\ 4 & (i = n) \end{cases} \tag{11}$$

$$u_i^{(1,2)} = \begin{cases} 5 & (i = 1) \\ 4 & (i = 2,\dots,n) \end{cases} \tag{12}$$

and

$$u_i^{(1,3)} = \begin{cases} 5 & (i = 1,\dots,n-1) \\ 6 & (i = n) \end{cases} \tag{13}$$

and for $(r = 1,2,3)$, let

$$u_i^{(k+1,r)} = \begin{cases} 5u_i^{(k,r)} & (i = 1,\dots,n-1) \\ 5u_i^{(k,r)} + 1 & (i = n) \end{cases} , \tag{14}$$

Then by (11) - (14), for $(\forall k \geq 0)$

$$u_i^{(k,1)} = \begin{cases} 3(5^{(k-1)}) & (i = 1,\dots,n-1) \\ \frac{17}{4}(5^{(k-1)}) - \frac{1}{4} & (i = n) \end{cases} , \tag{15}$$

$$u_i^{(k,2)} = \begin{cases} 5(5^{(k-1)}) & (i=1) \\ 4(5^{(k-1)}) & (i=2,...,n-1) \\ \frac{17}{4}(5^{(k-1)}) - \frac{1}{4} & (i=n) \end{cases}, \tag{16}$$

and

$$u_i^{(k,3)} = \begin{cases} 5(5^{(k-1)}) & (i=1,...,n-1) \\ \frac{25}{4}(5^{(k-1)}) - \frac{1}{4} & (i=n) \end{cases}, \tag{17}$$

Suppose, without any loss of generality, that

$$w_i^{(0,0)} \le h < 1 \quad (i=1,...,n) \tag{18}$$

Then by inductive argument it follows from (9) - (18) that for $(i=1,...,n)$ $(k \ge 0)$

$$w_i^{(k,1)} \le h^{u_i^{(k+1,1)}}, \quad w_i^{(k,2)} \le h^{u_i^{(k+1,2)}} \text{ and } w_i^{(k,3)} \le h^{u_i^{(k+1,3)}}, \tag{19}$$

whence by (17)-(19) and (4g),

$$w_i^{(k+1)} \le h^{5^{(k+1)}} \quad (i=1,...,n). \tag{20}$$

So, $(\forall k \ge 0)$, by (9) and (20),

$$w(X_i^{(k)}) \le (\frac{\beta}{\alpha})h^{5^{(k)}} \quad (i=1,...,n), \quad \alpha > 0. \tag{21}$$

Let

$$w^{(k)} = \max_{1 \le i \le n} \left\{ w(X_i^{(k)}) \right\} \tag{22}$$

Then by (21) and (22),

$$w^{(k)} \le (\frac{\beta}{\alpha})h^{5^k} \quad (\forall k \ge 0). \tag{23}$$

So, by (23), and the definition of *R*-factor in Ortega and Rheinboldt (1970), we have

$$R_5(w^{(k)}) = \lim_{k\to\infty}\sup\{(w^{(k)})^{1/(5^k)}\}$$
$$= \lim_{k\to\infty}\{(\frac{\beta}{\alpha})^{1/(5^k)} h\}$$
$$= h \quad < 1.$$

Therefore, it is proven that according to Alefeld and Herzberger (1983), Monsi and Wolfe (1988) and Ortega and Rheinboldt (1970), the order of convergence of IZSS1-5D is at least five or $O_R(IZSS1-5D, x_i^*) \geq 5$, $(i = 1,...,n)$.

4. RESULTS AND DISCUSSIONS

We use the Intlab V5.5 toolbox of Rump (1999) for Matlab R2012 to get the following results below. The algorithms IZSS1 and IZSS1-5D are run on five test polynomials where the stopping criterion used is $w^{(k)} \leq 10^{-10}$. The polynomials are as follows.

Test polynomial 1 : (Sham *et al.* (2013c))

$$p = \lambda^6 - 44\lambda^4 + 453\lambda^2 - 990$$

where

$$n = 6, \ a_1 = 5.4772, \ a_2 = -5.4772, \ a_3 = 3.3166, \ a_4 = 1.7321,$$
$$a_5 = -3.3166, \ a_6 = -1.7321.$$
$$b_i = 1 \ (i = 1,...,5)$$

Initial intervals:

$$X_1^{(0)} = [1, 2], \ X_2^{(0)} = [3, 4], \ X_3^{(0)} = [5, 6], \ X_4^{(0)} = [-2, -1],$$
$$X_5^{(0)} = [-4, -3], \ X_6^{(0)} = [-6, -5].$$

Test polynomial 2 : (Ehrlic (1967) , Alefeld and Herzberger (1983))

$$p = \lambda^5 - 35.6\lambda^4 + 482.86\lambda^3 - 3090.376\lambda^2 + 9197.7665\lambda - 9931.285$$

where

$$n=5,\ a_1=11.5,\ \ a_2=9.1,\ \ a_3=7.3,\ \ a_4=5.2,\ \ a_5=2.5,\ b_i=1\ (i=1,\ldots,4)$$

Initial intervals:

$$X_1^{(0)}=[-2.5,2.1],\ \ X_2^{(0)}=[2.2,4.5],\ \ X_3^{(0)}=[4.6,7.9],$$
$$X_4^{(0)}=[8.0,10.8],\ \ X_5^{(0)}=[10.9,13.1]$$

Test polynomial 3: (Sham *et al.* (2013c))

$$p=\lambda^4-10\lambda^3+35\lambda^2-50\lambda+24$$

where

$$n=4,\ a_1=1,\ \ a_2=2,\ \ a_3=3,\ \ a_4=4,\ b_i=1\ (i=1,\ldots,3)$$

Initial intervals:

$$X_1^{(0)}=[0.6,1.3],\ \ X_2^{(0)}=[1.6,2.3],\ \ X_3^{(0)}=[2.6,3.3],\ \ X_4^{(0)}=[3.6,4.3].$$

Test polynomial 4: (Alefeld and Herzberger 1983)

$$p=\lambda^9-398\lambda^7+45944\lambda^5-1778055\lambda^3+17863791\lambda$$

where

$$n=9,\ a_1=15,\ \ a_2=10,\ \ a_3=7,\ \ a_4=4,\ \ a_5=0,\ \ a_6=-4,\ \ a_7=-7,$$
$$a_8=-10,\ \ a_9=-15,\ b_i=1\ (i=1,\ldots,8)$$

Initial intervals:

$$X_1^{(0)}=[12.0,17.0],\ \ X_2^{(0)}=[8.6,11.2],\ \ X_3^{(0)}=[5.2,8.4],\ \ X_4^{(0)}=[2.4,5.0],$$
$$X_5^{(0)}=[-2.0,2.2], X_6^{(0)}=[-6.4,-2.9],$$
$$X_7^{(0)}=[-8.2,-6.5],\ \ X_8^{(0)}=[-11.8,-8.0],\ \ X_9^{(0)}=[-17.2,-13.5]$$

Test polynomial 5: (Alefeld and Herzberger (1983))

$$p=\lambda^5-30\lambda^4+311\lambda^3-1278\lambda^2+1551\lambda+630$$

where

$$n=5,\ a_1=0,\ \ a_2=3,\ \ a_3=6,\ \ a_4=9,\ \ a_5=12,\ b_i=1\ (i=1,\ldots,4)$$

Initial intervals:

$X_1^{(0)} = [1.9, 3.4], \quad X_2^{(0)} = [4.8, 5.9], \quad X_3^{(0)} = [6.5, 8.1],$

$X_4^{(0)} = [8.3, 9.8], \quad X_5^{(0)} = [10.7, 11.9].$

The tables show the results of all five test polynomials. Table 1 shows the comparison of the number of iteration and CPU time in seconds, between procedures ISS1 and IZSS1-5D.

TABLE 1: Number of Iterations and CPU Times

Polynomial	**n**	**IZSS1**		**IZSS1-5D**	
		No. of iterations	**CPU Times (seconds)**	**No. of iterations**	**CPU Times (seconds)**
1	6	2	0.32813	2	0.23438
2	5	2	0.25000	2	0.21875
3	4	3	0.21875	2	0.15625
4	9	3	0.71875	3	0.703125
5	5	2	0.21875	2	0.171875

TABLE 2: The width of final intervals at iteration *k* (Polynomial 3, *n* = 4)

Iteration k	**Interval width $w_i^{(k)}$**	**Methods**	
		IZSS1	**IZSS1-5D**
1	$w_1^{(1)}$	0.004440	0.003856
	$w_2^{(1)}$	0.095919	0.035329
	$w_3^{(1)}$	0.002974	0.002906
	$w_4^{(1)}$	0.001150	0.001121
2	$w_1^{(2)}$	3.456×10^{-11}	6.55×10^{-12}
	$w_2^{(2)}$	2.96×10^{-11}	2.13×10^{-12}
	$w_3^{(2)}$	5.00×10^{-14}	5.00×10^{-14}
	$w_4^{(2)}$	1.00×10^{-14}	1.00×10^{-14}
3	$w_1^{(3)}$	1.00×10^{-13}	(converged)
	$w_2^{(3)}$	3.96×10^{-12}	
	$w_3^{(3)}$	5.00×10^{-14}	
	$w_4^{(3)}$	1.00×10^{-14}	

For Table 2, we display the width of intervals at every iteration of the procedures IZSS1-5D and IZSS1 using test polynomial 3, before the algorithm stop at the stopping criterion, $w^{(k)} \leq 10^{-12}$. Table 1 shows that IZSS1-5D converges faster than IZSS1. Note that for test polynomials 3, the procedure IZSS1-5D requires less number of iterations than does procedure IZSS1 except for test polynomials 1, 2, 4 and 5 where the numbers of iterations of both procedures are the same. However, for these four test polynomials, the procedure IZSS1-5D consumes less CPU times compared to the IZSS1 procedure.

5. CONCLUSION

It has been proven that IZSS1-5D has a faster rate of convergence of at least five, whereas the *R*-order of convergence of IZSS1 is at least four (Rusli *et al.* 2011). Thus, we have this relationship $O_R(\text{IZSS1-5D}, x_i^*) > O_R(\text{IZSS1}, x_i^*) \quad (i = 1, ..., n)$. Our proposed procedure IZSS1-5D contributes to shorter CPU times and reduced number of iterations thus enhances the rate of convergence

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