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Chromaticity of a Family of K_4-Homeomorphs with Girth 9, II

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ABSTRACT

For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we investigate the chromaticity of one family of K_4-homeomorphs which has girth 9, and give sufficient and necessary condition for the graph in the family to be chromatically unique.

Keywords: Chromatic polynomial, Chromatically unique, K_4-homeomorphs.

1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G,) = P(H,)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

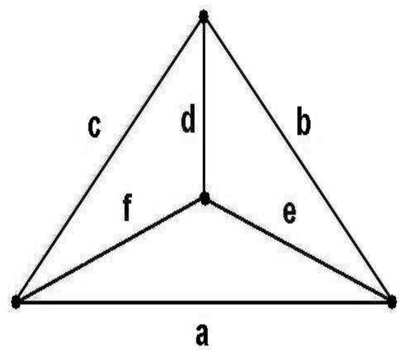

Figure 1: $K_4(a,b,c,d,e,f)$

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \le g \le 7$ has been studied by many authors (see Chen and Ouyang (1997), Li (1987), Peng (2004), Peng (2008), Peng (2012)). Also the chromaticity of K_4-homeomorphs with at least 2 paths of length 1 has been completely determined (Guo and Whitehead Jr. (1997), Li (1987), Peng and Liu (2002), Xu (1993)). Recently, Shi et al. (2012) studied the chromaticity of one family of K_4-homeomorphs with girth 8, i.e., $K_4(2,3,3,d,e,f)$. He then solved completely the chromaticity of K_4-homeomorphs with girth 8 (Shi (2011)). Ren (2002) has also completely determined the chromaticity of K_4-homeomorphs with exactly 3 paths of same length. Recently, Catada-Ghimire and Hasni (2014) investigated the chromaticity of K_4-homeomorphs with exactly 2 paths of length 2. The chromaticity of one family of K_4-homeomorphs with girth 9, that

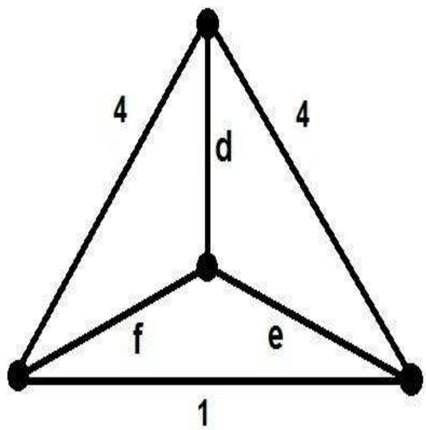

Figure 2: $K_4(1, 4, 4, d, e, f)$

is, the graph $K_4(2, 3, 4, d, e, f)$ has been studied by Karim and Lau (2014). Hence, to completely determine the chromaticity of K_4-homeomorphs with girth 9, there are only 5 more types to consider, that is, $K_4(1, 2, 6, d, e, f)$, $K_4(1, 3, 5, d, e, f)$, $K_4(1, 4, 4, d, e, f)$, $K_4(1, 2, c, 3, e, 3)$ and $K_4(1, 3, c, 2, e, 3)$. In this paper, we consider the chromaticity of one type of them, that is, the graph $K_4(1, 4, 4, d, e, f)$ (see Figure 2).

2. Preliminary Results

In this section, we give some known results used in the sequel.

Lemma 2.1. *Assume that G and H are $\chi-$equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see Koh and Teo (1990));*

(2) *G and H have the same girth and same number of cycles with length equal to their girth (see Xu (1991));*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see Chao and Zhao (1983));*

(4) *Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then*

(i) *min* $(a, b, c, d, e, f) =$ *min* (a', b', c', d', e', f') *and the number of times that this minimum occurs in the list* $\{a, b, c, d, e, f\}$ *is equal to the number of times that this minimum occurs in the list* $\{a', b', c', d', e', f'\}$ *(see Whitehead Jr. and Zhao (1984));*

(ii) *if* $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ *as multisets, then* $H \cong G$ *(see Li (1987)).*

Lemma 2.2. *(Karim and Lau (2014)) Let* K_4*-homeomorphs* $K_4(1, 4, 4, d, e, f)$ *and* $K_4(2, 3, 4, d', e', f')$ *be chromatically equivalent, then*
$K_4(1, 4, 4, 4, 2, 6) \sim K_4(2, 3, 4, 1, 7, 4)$, $K_4(1, 4, 4, 6, 2, 6) \sim K_4(2, 3, 4, 1, 5, 8)$.

Lemma 2.3. *(Aklan (2012)) Let* K_4*-homeomorphs* $K_4(1, 4, 4, d, e, f)$ *and* $K_4(1, 4, 4, d', e', f')$ *be chromatically equivalent, then*

$$K_4(1, 4, 4, i, i + 1, i + 5) \sim K_4(1, 4, 4, i + 2, i, i + 4).$$

where $i \geq 2$.

Lemma 2.4. *(Ren (2002)) Let* $G = K_4(a, b, c, d, e, f)$ *with exactly three of* a, b, c, d, e, f *are the same. Then* G *is not chromatically unique if and only if* G *is isomorphic to* $K_4(s, s, s - 2, 1, 2, s)$ *or* $K_4(s, s - 2, s, 2s - 2, 1, s)$ *or* $K_4(t, t, 1, 2t, t + 2, t)$ *or* $K_4(t, t, 1, 2t, t - 1, t)$ *or* $K_4(t, t + 1, t, 2t + 1, 1, t)$ *or* $K_4(1, t, 1, t + 1, 3, 1)$ *or* $K_4(1, 1, t, 2, t + 2, 1)$*, where* $s \geq 3$, $t \geq 2$.

Lemma 2.5. *(Catada-Ghimire and Hasni (2014)) A* K_4*-homeomorphic graph with exactly two path of length two is* χ*-unique if and only if it is not isomorphic to* $K_4(1, 2, 2, 4, 1, 1)$ *or* $K_4(4, 1, 2, 1, 2, 4)$ *or* $K_4(1, s + 2, 2, 1, 2, s)$ *or* $K_4(1, 2, 2, t + 2, t + 2, t)$ *or* $K_4(1, 2, 2, t, t + 1, t + 3)$ *or* $K_4(3, 2, 2, r, 1, 5)$ *or* $K_4(1, r, 2, 4, 2, 4)$ *or* $K_4(3, 2, 2, r, 1, r + 3)$ *or* $K_4(r + 2, 2, 2, 1, 4, r)$ *or* $K_4(r + 3, 2, 2, r, 1, 3)$ *or* $K_4(4, 2, 2, 1, r + 2, r)$ *or* $K_4(3, 4, 2, 4, 2, 6)$ *or* $K_4(3, 4, 2, 4, 2, 8)$ *or* $K_4(3, 4, 2, 8, 2, 4)$ *or* $K_4(7, 2, 2, 3, 4, 5)$ *or* $K_4(5, 2, 2, 3, 4, 7)$ *or* $K_4(8, 2, 2, 3, 4, 6)$ *or* $K_4(5, 2, 2, 9, 3, 4)$ *or* $K_4(5, 2, 2, 5, 3, 4)$*, where* $r \geq 3$, $s \geq 3$, $t \geq 3$.

3. Main Results

In this section, we present our main results. In the following, we only consider graphs with at most a path of length 1 and have girth 9.

Lemma 3.1. *If* G *is of type of* $K_4(1, 4, 4, d, e, f)$*, and* H *is of type of* $K_4(1, 3, 5, d', e', f')$*, then* G *is not chromatically equivalent to* H *except that*

$$K_4(1, 4, 4, 3, 5, 8) \sim K_4(1, 3, 5, 5, 7, 4),$$

$$K_4(1,4,4,6,3,7) \sim K_4(1,3,5,4,4,8),$$

$$K_4(1,4,4,6,3,8) \sim K_4(1,3,5,4,9,4),$$

$$K_4(1,4,4,6,2,6) \sim K_4(1,3,5,2,4,8).$$

Proof. Let G and H be two graphs such that $G \cong K_4(1,4,4,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$. Let

$$\begin{aligned} Q(K_4(a,b,c,d,e,f)) &= -(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+s^{a+d}+s^{b+f}+ \\ &\quad s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}. \end{aligned}$$

Let $s = 1-\lambda$ and x is the number of edges in G. From Shi et al. (2012), we have the chromatic polynomial of K_4-homeomorphs $K_4(a,b,c,d,e,f)$ is as follows:

$$P(K_4(a,b,c,d,e,f) = (-1)^{x-1}\frac{s}{(s-1)^2}\Big[(s^2+3s+2)+Q(K_4(a,b,c,d,e,f))-s^{x-1})\Big].$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively.

As $G \cong K_4(1,4,4,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$, then

$$\begin{aligned} Q(G) &= -(s+1)(s+s^4+s^4+s^d+s^e+s^f)+s^{d+1}+s^{f+4}+ \\ &\quad s^{e+4}+s^{e+5}+s^{d+8}+s^{f+5}+s^{d+e+f}. \\ Q(H) &= -(s+1)(s+s^3+s^5+s^{d'}+s^{e'}+s^{f'})+s^{d'+1}+s^{f'+3}+ \\ &\quad s^{e'+5}+s^{e'+4}+s^{d'+8}+s^{f'+6}+s^{d'+e'+f'}. \end{aligned}$$

By symmetry of $K_4(1,4,4,d,e,f)$, we can assume that $e \le f$. From Lemma 2.1 (1),

$$d+e+f = d'+e'+f' \tag{1}$$

$Q(G) = Q(H)$ yields

$$\begin{aligned} Q_1(G) &= -s^4 - s^5 - s^d - s^e - s^f - s^{e+1} - s^{f+1} + \\ &\quad s^{d+8} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}. \\ Q_1(H) &= -s^3 - s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + \\ &\quad s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}. \end{aligned}$$

Comparing the l.r.p in $Q_1(G)$ and the l.r.p in $Q_1(H)$, we have $d = 3$ or $e = 2$ or $e = 3$. There are three cases to be considered.

Case A $d = 3$. We obtain the following after simplification.

$$Q_2(G) = -s^4 - s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_2(H) = -s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}.$$

By considering the h.r.p in $Q_2(G)$, we have the h.r.p in $Q_2(G)$ is 11 or $f+5$. The h.r.p in $Q_2(H)$ is $d' + 8$ or $e' + 5$ or $f' + 6$. There are two cases to be considered.

Case 1 The h.r.p in $Q_2(G)$ is 11. There are three cases to be considered.

Case 1.1 If $d' + 8 = 11$, then $d' = 3$. We have the following after simplification.

$$Q_3(G) = -s^4 - s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_3(H) = -s^3 - s^6 - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}.$$

Comparing the h.r.p in $Q_3(G)$ and the h.r.p in $Q_3(H)$, we have $f+5 = e'+5$ or $f + 5 = f' + 6$.

If $f + 5 = e' + 5$, then $f = e'$. By Equation (1), we get $e = f'$, then

$Q_3(G) \neq Q_3(H)$, a contradiction.

If $f+5 = f'+6$, then $f = f'+1$. By Equation (1), we get $e+1 = e'$. We obtain the following after simplification.

$$Q_4(G) = -s^4 - s^5 - s^e - s^{f+1} + s^{e+4} + s^{f+4},$$

$$Q_4(H) = -s^3 - s^6 - s^{e+2} - s^{f-1} + s^{e+6} + s^{f+2}.$$

Then we have $e = 3$, $f = 5$, $e' = 4$ and $f' = 4$. Thus, $G \cong H$.

Case 1.2 If $e'+5 = 11$, then $e' = 6$. We have the following after simplification.

$$Q_5(G) = -s^4 - s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_5(H) = -s^6 - s^6 - s^7 - s^{d'} - s^{f'} - s^{f'+1} + s^{10} + s^{d'+8} + s^{f'+3} + s^{f'+6}.$$

Comparing the l.r.p in $Q_5(G)$ and the l.r.p in $Q_5(H)$, we have $d' = 4$ or $f' = 4$ or $f' = 3$.

Case 1.2.1 $d' = 4$. We obtain the following after simplification.

$$Q_6(G) = -s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_6(H) = -s^6 - s^6 - s^7 - s^{f'} - s^{f'+1} + s^{10} + s^{12} + s^{f'+3} + s^{f'+6}.$$

Comparing the l.r.p in $Q_6(G)$ and the l.r.p in $Q_6(H)$, we have $f' = 4$ or $f' = 5$. It is easy to handle these cases in the same fashion as in Case 1.1, and we obtain $Q_6(G) \neq Q_6(H)$, a contradiction.

Case 1.2.2 $f' = 4$. We obtain the following after simplification.

$$Q_7(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_7(H) = -s^6 - s^6 - s^{d'} + s^{10} + s^{10} + s^{d'+8}.$$

Comparing the h.r.p in $Q_7(G)$ and the h.r.p in $Q_7(H)$, we have $f+5 = d'+8$. So $f = d'+3$, then we get $Q_7(G) \neq Q_7(H)$, a contradiction.

Case 1.2.3 $f' = 3$. We obtain the following after simplification.

$$Q_8(G) = -s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_8(H) = -s^3 - s^6 - s^7 - s^{d'} + s^9 + s^{10} + s^{d'+8}.$$

Similar to Case 1.2.2, we get $Q_8(G) \neq Q_8(H)$, a contradiction.

Case 1.3 If $f' + 6 = 11$, then $f' = 5$. We have the following after simplification.

$$Q_9(G) = -s^4 - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_9(H) = -2s^6 - s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}.$$

Comparing the l.r.p in $Q_9(G)$ and the l.r.p in $Q_9(H)$, we have $d' = 4$ or $e' = 4$ or $e' = 3$.

Case 1.3.1 If $d' = 4$. We obtain the following after simplification.

$$Q_{10}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5},$$

$$Q_{10}(H) = -2s^6 - s^{e'} - s^{e'+1} + s^8 + s^{12} + s^{e'+4} + s^{e'+5}.$$

Comparing the h.r.p in $Q_{10}(G)$ and the h.r.p in $Q_{10}(H)$, we have $f+5=12$ or $f+5=e'+5$.

If $f+5=12$, so $f=7$. From Equation (1), we get $e+1=e'$. We then obtain $Q_{10}(G)\neq Q_{10}(H)$, a contradiction.

If $f+5=e'+5$, so $f=e'$. From Equation (1), we get $e=6$. We then obtain $Q_{10}(G)\neq Q_{10}(H)$, a contradiction.

Case 1.3.2 If $e'=4$. We obtain the following after simplification.

$$Q_{11}(G)=-s^e-s^f-s^{e+1}-s^{f+1}+s^{e+4}+s^{e+5}+s^{f+4}+s^{f+5},$$

$$Q_{11}(H)=-s^5-2s^6-s^{d'}+2s^8+s^9+s^{d'+8}.$$

Comparing the h.r.p in $Q_{11}(G)$ and the h.r.p in $Q_{11}(H)$, we have $f+5=d'+8$, so $f=d'+3$. We get $Q_{11}(G)\neq Q_{11}(H)$, a contradiction.

Case 1.3.3 If $e'=3$. We obtain the following after simplification.

$$Q_{12}(G)=-s^e-s^f-s^{e+1}-s^{f+1}+s^{e+4}+s^{e+5}+s^{f+4}+s^{f+5},$$

$$Q_{12}(H)=-s^3-2s^6-s^{d'}+s^7+2s^8+s^{d'+8}.$$

Similar to Case 1.3.2, we get $Q_{12}(G)\neq Q_{12}(H)$, a contradiction.

Case 2 The h.r.p in $Q_2(G)$ is $f+5$. There are three cases to be considered.

Case 2.1 $f+5=d'+8$. From $Q_2(G)$ and $Q_2(H)$, we obtain the following after simplification.

$$Q_{13}(G) = -s^4 - s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4},$$

$$Q_{13}(H) = -s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}.$$

Comparing the l.r.p in $Q_{13}(G)$ and the l.r.p in $Q_{13}(H)$, we have $d'=4$ or $e'=4$ or $f'=4$ or $e'=3$ or $f'=3$.

Case 2.1.1 $d'=4$. From Equation (1), we get $f=d'+3$, so $f=7$. We obtain the following after simplification.

$$Q_{14}(G) = -s^5 - s^7 - s^8 - s^e - s^{e+1} + 2s^{11} + s^{e+4} + s^{e+5},$$

$$Q_{14}(H) = -s^6 - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}.$$

Comparing the l.r.p in $Q_{14}(G)$ and the l.r.p in $Q_{14}(H)$, we have $e'=5$ or $f'=5$ or $e'=4$ or $f'=4$.

If $e'=5$, by Equation (1), we get $e+1=f'$, then $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

If $f'=5$, by Equation (1), we get $e+1=e'$, then $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

If $e'=4$, by Equation (1), we get $e+2=f'$, then $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

If $f'=4$, by Equation (1), we get $e+2=e'$, then $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

Case 2.1.2 $e'=4$. From Equation (1), we get $f=d'+3$, so $f=7$. We obtain the following after simplification.

$$Q_{15}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4},$$

$$Q_{15}(H) = -s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^8 + s^9 + s^{f'+3} + s^{f'+6}.$$

Comparing the h.r.p in $Q_{15}(G)$ and the h.r.p in $Q_{15}(H)$, we have $f'+6=11$ or $f'+6=f+4$ or $e+5=f'+6$.

If $f'+6=11$, so $f'=5$. By Equation (1), we get $e=3$, then $Q_{15}(G) \neq Q_{15}(H)$, a contradiction.

If $f'+6=f+4$, so $f=f'+2$. By Equation (1), we get $e+1=d'$, then $Q_{15}(G) \neq Q_{15}(H)$, a contradiction.

If $f'+6=e+5$, so $f'+1=e$. By Equation (1), $f=d'$, and thus $Q_{15}(G) \neq Q_{15}(H)$, a contradiction.

Case 2.1.3 $f'=4$. We obtain the following after simplification.

$$Q_{16}(G) = -s^{e} - s^{f} - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4},$$

$$Q_{16}(H) = -s^{6} - s^{d'} - s^{e'} - s^{e'+1} + s^{7} + s^{10} + s^{e'+4} + s^{e'+5}.$$

Comparing the h.r.p in $Q_{16}(G)$ and the h.r.p in $Q_{16}(H)$, we have $e+5=10$ or $f+4=10$ or $e'+5=11$ or $e'+5=e+5$ or $e'+5=f+4$.

Case 2.1.3.1 $e+5=10$, so $e=5$, by Equation (1), we get $e'=7$. Note that $f=d'+3$. We obtain the following after simplification.

$$Q_{17}(G) = -s^{5} - s^{f} - s^{f+1} + s^{9} + s^{f+4},\ Q_{17}(H) = -s^{f-3} - s^{8} + s^{12}.$$

Thus, we have $f=8$ and $d'=5$. We then obtain the solution where G is isomorphic to $K_4(1,4,4,3,5,8)$ and H is isomorphic to $K_4(1,3,5,5,7,4)$. That is

$$K_4(1,4,4,3,5,8) \sim K_4(1,3,5,5,7,4).$$

Case 2.1.3.2 $f+4=10$, so $f=6$, and from $f=d'+3$, we have $d'=3$. By Equation (1), we obtain $e+2=e'$. We get $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case 2.1.3.3 $e'+5=11$, so $e'=6$, by Equation (1), we obtain $e=4$. We then get $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case 2.1.3.4 $e+5=e'+5$, so $e=e'$. We then get $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case 2.1.3.5 $f+4=e'+5$, so $f=e'+1$. We obtain the following after simplification.

$$Q_{18}(G)=-s^{e}-s^{e+1}-s^{f+1}+s^{11}+s^{e+4}+s^{e+5},$$

$$Q_{18}(H)=-s^{6}-s^{d'}-s^{e'}+s^{7}+s^{10}+s^{f+3}.$$

Comparing the h.r.p in $Q_{18}(G)$ and the h.r.p in $Q_{18}(H)$, we have $f+3=11$ or $e+5=10$ or $e+5=f+3$.

If $f+3=11$, so $f=8$. After simplification of $Q_{18}(G)$ and $Q_{18}(H)$, we have $e=d'=5$ and $e'=7$. We then obtain the solution where G is isomorphic to $K_4(1,4,4,3,5,8)$ and H is isomorphic to $K_4(1,3,5,5,7,4)$, that is

$$K_4(1,4,4,3,5,8)\sim K_4(1,3,5,5,7,4).$$

If $e+5=10$, so $e=5$. We then obtain the solution where G is isomorphic to $K_4(1,4,4,3,5,8)$ and H is isomorphic to $K_4(1,3,5,5,7,4)$, that is

$$K_4(1,4,4,3,5,8)\sim K_4(1,3,5,5,7,4).$$

If $e+5=f+3$, so $e+2=f$. After simplification, we obtain $Q_{18}(G)\neq Q_{18}(H)$, a contradiction.

Case 2.1.4 $e'=3$. We obtain the following after simplification.

$$Q_{19}(G)=-s^{5}-s^{e}-s^{f}-s^{e+1}-s^{f+1}+s^{11}+s^{e+4}+s^{e+5}+s^{f+4},$$

$$Q_{19}(H)=-s^{3}-s^{6}-s^{d'}-s^{f'}-s^{f'+1}+s^{7}+s^{8}+s^{f'+3}+s^{f'+6}.$$

Comparing the h.r.p in $Q_{19}(G)$ and the h.r.p in $Q_{19}(H)$, we obtain $f'+6=11$ or $e+5=f'+6$ or $f+4=f'+6$.

Case 2.1.4.1 $f'+6=11$, so $f'=5$. From Equation (1), $e=2$. After simplification, we get $Q_{19}(G)\neq Q_{19}(H)$, a contradiction.

Case 2.1.4.2 $e+5=f'+6$, so $e=f'+1$. Similarly, we get $Q_{19}(G)\neq Q_{19}(H)$, a contradiction.

Case 2.1.4.3 $f+4=f'+6$, so $f=f'+2$. We obtain the following after simplification.

$$Q_{20}(G) = -s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5},$$

$$Q_{20}(H) = -s^3 - s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^7 + s^8 + s^{f+1}.$$

Comparing the h.r.p in $Q_{20}(G)$ and the h.r.p in $Q_{20}(H)$, we obtain $f+1 = 11$ or $e+5 = 8$ or $e+5 = f+1$.

If $f+1 = 11$, so $f = 10$. Then $d' = 3$ and by Equation (1), we get $e = 5$. It can be checked that $Q_{20}(G) \neq Q_{20}(H)$.

If $e+5 = 8$, so $e = 3$. By Equation (1), we get $f' = 6$ and then $f = 8$. It can be checked that $Q_{20}(G) \neq Q_{20}(H)$.

If $e+5 = f+1$, so $e+4 = f$, then we get $Q_{20}(G) \neq Q_{20}(H)$.

Case 2.1.5 $f' = 3$. We obtain the following after simplification.

$$Q_{21}(G) = -s^5 - s^e - s^f - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4},$$

$$Q_{21}(H) = -s^3 - s^{d'} - s^{e'} - s^{e'+1} + s^9 + s^{e'+4} + s^{e'+5}.$$

If $e'+5 = 11$, then $e' = 6$. From Equation (1), $e = 3$. Similar to the cases above, we obtain $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.

If $e'+5 = e+5$, then $e' = e$. Similar to the cases above, we obtain $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.

If $e'+5 = f+4$, then $e'+1 = f$. Similar to the cases above, we obtain $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.

Case 2.2 $f+5 = e'+5$. We obtain the following after simplification.

$$Q_{22}(G) = -s^4 - s^5 - s^e - s^{e+1} + s^{11} + s^{e+4} + s^{e+5},$$

$$Q_{22}(H) = -s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{f'+3} + s^{f'+6}.$$

Comparing the l.r.p in $Q_{22}(G)$ and the l.r.p in $Q_{22}(H)$, we obtain $d' = 4$ or $f' = 4$ or $f' = 3$.

If $d' = 4$, by Equation (1), $e = f'$. Similar to the cases above, we obtain

$Q_{22}(G) \neq Q_{22}(H)$, a contradiction.

If $f' = 4$, by Equation (1), $e = d' + 1$. Similar to the cases above, we obtain $Q_{22}(G) \neq Q_{22}(H)$, a contradiction.

If $f' = 3$, by Equation (1), $e = d'$. Similar to the cases above, we obtain $Q_{22}(G) \neq Q_{22}(H)$, a contradiction.

Case 2.3 $f + 5 = f' + 6$, so $f = f' + 1$. We obtain the following after simplification.

$$Q_{23}(G) = -s^4 - s^5 - s^e - s^{e+1} - s^{f+1} + s^{11} + s^{e+4} + s^{e+5} + s^{f+4},$$

$$Q_{23}(H) = -s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} + s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3}.$$

Comparing the l.r.p in $Q_{23}(G)$ and the l.r.p in $Q_{23}(H)$, we obtain $d' = 4$ or $f' = 4$ or $e' = 4$ or $e' = 3$.

Case 2.3.1 $d' = 4$. From Equation (1), $e = e'$. We can see that $Q_{23}(G) \neq Q_{23}(H)$, a contradiction.

Case 2.3.2 $e' = 4$. From Equation (1), $e = d'$. We can see that $G \cong H$.

Case 2.3.3 $f' = 4$. So $f = 5$. We obtain the following after simplification.

$$Q_{24}(G) = -s^5 - s^e - s^{e+1} + s^9 + s^{11} + s^{e+4} + s^{e+5},$$

$$Q_{24}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^7 + s^{d'+8} + s^{e'+4} + s^{e'+5}.$$

Comparing the l.r.p in $Q_{24}(G)$ and the l.r.p in $Q_{24}(H)$, we obtain $d' = 5$ or $e' = 5$ or $e' = 4$.

Case 2.3.3.1 $d' = 5$. From Equation (1), $e = e' + 1$. After simplifying $Q_{24}(G)$ and $Q_{24}(H)$, we have $e = 8$ and $e' = 7$. Thus, $G \cong K_4(1, 4, 4, 3, 8, 5)$ and $H \cong K_4(1, 3, 5, 5, 7, 4)$ and hence, $K_4(1, 4, 4, 3, 8, 5) \sim K_4(1, 3, 5, 5, 7, 4)$. But this is a contradiction since $e \leq f$.

Case 2.3.3.2 $e' = 5$. From Equation (1), $e = d' + 1$. We can see that $Q_{24}(G) \neq Q_{24}(H)$, a contradiction.

Case 2.3.3.3 $e' = 4$. From Equation (1), $e = d'$. After simplifying $Q_{24}(G)$ and $Q_{24}(H)$, we have $e = d' = 3$. Thus, $G \cong K_4(1, 4, 4, 3, 3, 5)$ and $H \cong$

$K_4(1,3,5,3,4,4)$ and hence, $G \cong H$.

Case 2.3.4 $e' = 3$. We can see that $Q_{23}(G) \neq Q_{23}(H)$, a contradiction.

Case B $e = 3$. We obtain the following after simplification.

$Q_{25}(G) = -2s^4 - s^5 - s^d - s^f - s^{f+1} + s^7 + s^8 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{25}(H) = -s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Consider the term $-2s^4$ in $Q_{25}(G)$. Since $-2s^4$ in $Q_{25}(G)$ cannot be cancelled by any positive term in $Q_{25}(G)$, then it must be equal to two terms in $Q_{25}(H)$. Since $e' + f' \geq 8$, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$ or $d' = e' + 1 = 4$ or $d' = f' + 1 = 4$.

Case 1 $d' = e' = 4$. We obtain the following after simplification.

$Q_{26}(G) = -s^d - s^f - s^{f+1} + s^7 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{26}(H) = -s^6 - s^{f'} - s^{f'+1} + s^9 + s^{12} + s^{f'+3} + s^{f'+6}$.

Comparing the h.r.p in $Q_{26}(G)$ and the h.r.p in $Q_{26}(H)$, we obtain $d+8 = f'+6$ or $d+8 = 12$ or $f+5 = f'+6$ or $f+5 = 12$.

Case 1.1 $d+8 = f'+6$. So $d+2 = f'$. From Equation (1), $f = 7$. After simplifying, we obtain $d = 6$ and $f' = 8$. Therefore, $G \cong K_4(1,4,4,6,3,7)$ and $H \cong K_4(1,3,5,4,4,8)$ and hence,

$$K_4(1,4,4,6,3,7) \sim K_4(1,3,5,4,4,8).$$

Case 1.2 $d+8 = 12$. So $d = 4$. From Equation (1), $f = f'+1$. After simplifying, we obtain that $f = 5$ and $f' = 4$. Therefore $G \cong K_4(1,4,4,4,3,5)$ and $H \cong K_4(1,3,5,4,4,4)$, and hence $G \cong H$.

$Q_{27}(G) = -s^d - s^f - s^{f+1} + s^8 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{27}(H) = -s^6 - s^{e'} - s^{e'+1} + s^{10} + s^{12} + s^{e'+4} + s^{e'+5}$.

$$K_4(1,4,4,6,3,8) \sim K_4(1,3,5,4,9,4).$$

$Q_{28}(G) = -s^d - s^f - s^{f+1} + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{28}(H) = -s^5 - s^6 - s^{d'} + s^9 + s^{10} + s^{d'+8}$.

$Q_{29}(G) = -s^5 - s^d - s^f - s^{f+1} + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{29}(H) = -s^3 - s^6 - s^{f'} - s^{f'+1} + s^{12} + s^{f'+3} + s^{f'+6}$.

$Q_{30}(G) = -s^5 - s^d - s^f - s^{f+1} + s^7 + s^8 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{30}(H) = -s^3 - s^{e'} - s^{e'+1} + s^9 + s^{12} + s^{e'+4} + s^{e'+5}$.

$Q_{31}(G) = -s^2 - s^4 - s^5 - s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{31}(H) = -s^6 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

$Q_{32}(G) = -s^4 - s^5 - s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{32}(H) = -s^6 - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

$Q_{33}(G) = -s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{33}(H) = -s^6 - s^{f'} - s^{f'+1} + s^8 + s^9 + s^{10} + s^{f'+3} + s^{f'+6}$.

$$K_4(1,4,4,6,2,6) \sim K_4(1,3,5,2,4,8).$$

$Q_{34}(G) = -s^d - s^f - s^{f+1} + s^6 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{34}(H) = -s^6 - s^{e'} - s^{e'+1} + 2s^{10} + s^{e'+4} + s^{e'+5}$.

$Q_{35}(G) = -s^5 - s^d - s^f - s^{f+1} + s^7 + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{35}(H) = -s^3 - s^6 - s^{e'} - s^{e'+1} + s^9 + s^{10} + s^{e'+4} + s^{e'+5}$.

$Q_{36}(G) = -s^4 - s^5 - s^d - s^f - s^{f+1} + s^{d+8} + s^{f+4} + s^{f+5}$,

$Q_{36}(H) = -s^3 - s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

$Q_{37}(G) = -s^4 - s^5 - s^f - s^{f+1} + s^{11} + s^{f+4} + s^{f+5}$,

$$Q_{37}(H) = -s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{f'+3} + s^{f'+6}.$$

$$Q_{38}(G) = -s^4 - s^5 - s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5},$$

$$Q_{38}(H) = -s^3 - s^6 - s^{d'} - s^{e'} - s^{e'+1} + s^5 + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}.$$

$$Q_{39}(G) = -s^4 - s^5 - s^f - s^{f+1} + s^6 + s^7 + s^{11} + s^{f+4} + s^{f+5},$$

$$Q_{39}(H) = -s^6 - s^{d'} - s^{e'} - s^{e'+1} + s^5 + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}.$$

$$K_4(1,4,4,3,5,8) \sim K_4(1,3,5,5,7,4),$$

$$K_4(1,4,4,6,3,7) \sim K_4(1,3,5,4,4,8),$$

$$K_4(1,4,4,6,3,8) \sim K_4(1,3,5,4,9,4),$$

$$K_4(1,4,4,6,2,6) \sim K_4(1,3,5,2,4,8).$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *If G is of type of $K_4(1,4,4,d,e,f)$, and H is of type of $K_4(1,2,6,d',e',f')$, then G is not chromatically equivalent to H except that*

$$K_4(1,4,4,2,3,7) \sim K_4(1,2,6,4,4,4).$$

Proof. Let G and H be two graphs such that $G \cong K_4(1,4,4,d,e,f)$ and $H \cong K_4(1,2,6,d',e',f')$. Then

$$\begin{aligned} Q(G) &= -(s+1)(s+s^4+s^4+s^d+s^e+s^f)+s^{d+1}+s^{f+4}+ \\ &\quad s^{e+4}+s^{e+5}+s^{d+8}+s^{f+5}+s^{d+e+f}. \\ Q(H) &= -(s+1)(s+s^2+s^6+s^{d'}+s^{e'}+s^{f'})+s^{d'+1}+s^{f'+2}+ \\ &\quad s^{e'+6}+s^{e'+3}+s^{d'+8}+s^{f'+7}+s^{d'+e'+f'}. \end{aligned}$$

$Q(G) = Q(H)$ and from Equation (1) of Lemma 3.1 yield

$$\begin{aligned} Q_1(G) &= -2s^4 - 2s^5 - s^d - s^e - s^f - s^{e+1} - s^{f+1} + \\ & s^{d+8} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}. \\ Q_1(H) &= -s^2 - s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + \\ & s^{d'+8} + s^{e'+3} + s^{e'+6} + s^{f'+2} + s^{f'+7}. \end{aligned}$$

By symmetry of $K_4(1,4,4,d,e,f)$, we can assume that $e \leq f$. From Lemma 2.1 (1),

$$d + e + f = d' + e' + f' \tag{2}$$

Note that min $\{d, e\} = 2$. So, there are two cases to be considered.

Case A $d = 2$. From $d + e \geq 5$ and $e \leq f$, we have $3 \leq e \leq f$. We obtain the following after simplification.

$Q_2(G) = -2s^4 - 2s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{10} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}$.

$Q_2(H) = -s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{e'+3} + s^{e'+6} + s^{f'+2} + s^{f'+7}$.

The h.r.p in $Q_2(G)$ is 10 or $f + 5$.//

Case 1 $10 \geq f + 5$. Consider the h.r.p in $Q_2(H)$, so we have $e' + 6 = 10$ or $f' + 7 = 10$ or $d' + 8 = 10$.

Case 1.1 $e' + 6 = 10$. So $e' = 4$. We obtain the following after simplification.

$Q_3(G) = -s^4 - s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}$.

$Q_3(H) = -s^3 - s^6 - s^{d'} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{f'+2} + s^{f'+7}$.

Consider the h.r.p in $Q_3(G)$ and the h.r.p in $Q_3(H)$, we have $d' + 8 = f + 5$

or $f' + 7 = f + 5$.

Case 1.1.1 $d' + 8 = f + 5$. So $d' + 3 = f$. By Equation (2), $e + 1 = f'$. Cancelling the equal terms in $Q_3(G)$ and $Q_3(H)$ resulting the following.

$$Q_4(G) = -s^4 - s^5 - s^e - s^f - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4}.$$

$$Q_4(H) = -s^3 - s^6 - s^{e+2} - s^{f-3} + s^{e+3} + s^{e+8}.$$

After simplification, we obtain $e = 3$, $f = 7$, $d' = 4$ and $f' = 4$. Therefore, $G \cong K_4(1, 4, 4, 2, 3, 7)$ and $H \cong K_4(1, 2, 6, 4, 4, 4)$. Hence,

$$K_4(1, 4, 4, 2, 3, 7) \sim K_4(1, 2, 6, 4, 4, 4).$$

Case 1.1.2 $f' + 7 = f + 5$. So $f' + 2 = f$. By Equation (2), $e = d'$. Cancelling the equal terms in $Q_3(G)$ and $Q_3(H)$ resulting the following.

$$Q_5(G) = -s^4 - s^5 - s^{e+1} - s^f - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4}.$$

$$Q_5(H) = -s^3 - s^6 - s^{f-2} - s^{f-1} + s^f + s^{e+8}.$$

Consider the h.r.p in $Q_5(G)$ and the h.r.p in $Q_5(H)$, we have $f + 4 = e + 8$ or $e + 5 = f$. If $f = e + 4$, we obtain $e = 2$, a contradiction. If $f = e + 5$, we obtain that $Q_5(G) \neq Q_5(H)$, also a contradiction.

Case 1.2 $f' + 7 = 10$. So $f' = 3$. We obtain the following after simplification.

$$Q_6(G) = -s^4 - 2s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}.$$

$$Q_6(H) = -2s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{e'+1} + s^5 + s^{d'+8} + s^{e'+3} + s^{e'+6}.$$

Consider the h.r.p in $Q_6(G)$ and the h.r.p in $Q_6(H)$, we have $f + 5 = e' + 6$

or $f + 5 = d' + 8$.

Case 1.2.1 $f + 5 = e' + 6$. So $f = e' + 1$. By Equation (2), $e = d'$. We obtain the following after simplification.//

$$Q_7(G) = -s^4 - 2s^5 - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4}.$$

$$Q_7(H) = -2s^3 - s^6 - s^7 - s^{f-1} + s^5 + s^{e+8} + s^{f+2}.$$

The term $-2s^3$ is in $Q_7(H)$ but not in $Q_7(G)$, a contradiction.

Case 1.2.2 $f + 5 = d' + 8$. So $f = d' + 3$. By Equation (2), $e + 2 = e'$. Similar to Case 1.2.1, we obtain that $Q_6(G) \neq Q_6(H)$, a contradiction.

Case 1.3 $d' + 8 = 10$. So $d' = 2$.

$$Q_8(G) = -2s^4 - 2s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+4} + s^{f+5}.$$

$$Q_8(H) = -s^2 - s^3 - s^6 - s^7 - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{e'+3} + s^{e'+6} + s^{f'+2} + s^{f'+7}.$$

As $e \geq 3$, the highest terms in $Q_8(G)$ and $Q_8(H)$ are not equal, a contradiction.

Case 2 $10 \leq f + 5$. Consider the h.r.p in $Q_2(H)$, so we have $e' + 6 = f + 5$ or $f' + 7 = f + 5$ or $d' + 8 = f + 5$.

Case 2.1 $e' + 6 = f + 5$. So $e' + 1 = f$. Cancelling the equal terms in $Q_2(G)$ and $Q_2(H)$ yields the following.

$$Q_9(G) = -2s^4 - 2s^5 - s^e - s^{e+1} - s^{f+1} + s^{10} + s^{e+4} + s^{e+5} + s^{f+4}.$$

$$Q_9(H) = -s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{e'+3} + s^{f'+2} + s^{f'+7}.$$

Consider the term $-2s^4$ in $Q_9(G)$. Since $Q_9(G) = Q_9(H)$, there are two terms in $Q_9(H)$ equal to $-2s^4$. So we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$ or $d' = f' + 1 = 4$.

Case 2.1.1 $d' = e' = 4$. So $f = 5$. By Equation (2), $e = f' + 1$. We obtain the following after simplification.

$Q_{10}(G) = -2s^5 - s^{e+1} + s^9 + s^{10} + s^{e+4} + s^{e+5}$, $Q_{10}(H) = -s^3 - s^{e-1} + s^{12} + s^{e+1} + s^{e+6}$.

Since $-s^3$ is in $Q_{10}(H)$ but not in $Q_{10}(G)$, this is a contradiction.

Case 2.1.2 $d' = f' = 4$. So $e = 5$. By Equation (2), $e' + 1 = f$. Similar to Case 2.1.1, we obtain a contradiction.

Case 2.1.3 $e' = f' = 4$. So $f = 5$. By Equation (2), $e = d' + 1$. Similar to Case 2.1.1, we obtain a contradiction.

Case 2.1.4 $d' = f' + 1 = 4$. So $f' = 3$. By Equation (2), $e = 4$. Similar to Case 2.1.1, we obtain a contradiction.

Case 2.2 $f' + 7 = f + 5$. So $f' + 2 = f$. Cancelling the equal terms in $Q_2(G)$ and $Q_2(H)$ yields the following.

$Q_{11}(G) = -2s^4 - 2s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{10} + s^{e+4} + s^{e+5} + s^{f+4}$.

$Q_{11}(H) = -s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{e'+3} + s^{e'+6} + s^{f'+2}$.

Consider the term $-2s^4$ in $Q_{11}(G)$. For the same reason as in Case 2.1, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$ or $d' = e' + 1 = 4$ or

$d' = f' + 1 = 4$.

Case 2.2.1 $d' = e' = 4$. So $f = f' + 2$. By Equation (2), $e = 4$. We obtain the following after simplification.

$Q_{12}(G) = -s^4 - 2s^5 - s^f - s^{f+1} + s^8 + s^9 + s^{f+4}$, $Q_{12}(H) = -s^3 - s^6 - s^{f-2} - s^{f-1} + s^{12} + s^f$.

The term $-2s^5$ is in $Q_{12}(G)$ but not in $Q_{12}(H)$, a contradiction.

Case 2.2.2 $d' = f' = 4$. So $f = f' + 2$. By Equation (2), $e = e'$. Similar to Case 2.2.1, we obtain a contradiction.

Case 2.2.3 $e' = f' = 4$. So $f = f' + 2$. By Equation (2), $e = d'$. Similar to Case 2.2.1, we obtain a contradiction.

Case 2.2.4 $d' = e' + 1 = 4$. So $e' = 3$. By Equation (2), $e = 3$. Similar to Case 2.2.1, we obtain a contradiction.

Case 2.2.5 $d' = f' + 1 = 4$. So $f' = 3$ and $f = 5$. By Equation (2), $e = e'$. Similar to Case 2.2.1, we obtain a contradiction.

Case 2.3 $d' + 8 = f + 5$. So $d' + 3 = f$. Cancelling the equal terms in $Q_2(G)$ and $Q_2(H)$ yields the following.

$Q_{13}(G) = -2s^4 - 2s^5 - s^e - s^{e+1} - s^f - s^{f+1} + s^{10} + s^{e+4} + s^{e+5} + s^{f+4}$.

$Q_{13}(H) = -s^3 - s^6 - s^7 - s^{d'} - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{e'+3} + s^{e'+6} + s^{f'+2} + s^{f'+7}$.

Consider the term $-2s^4$ in $Q_{13}(G)$. For the same reason as in Case 2.1, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$ or $d' = e' + 1 = 4$ or

$d' = f' + 1 = 4$.

Case 2.3.1 $d' = e' = 4$. So $f = 7$. By Equation (2), $e + 1 = f'$. We obtain the following after simplification.

$$Q_{14}(G) = -s^5 - s^8 - s^e - s^{e+1} + s^{11} + s^{e+4} + s^{e+5}.$$

$$Q_{14}(H) = -s^3 - s^6 - s^{e+1} - s^{e+2} + s^7 + s^{e+3} + s^{e+8}.$$

Thus $e = 3$ and $f' = 4$. So $G \cong K_4(1, 4, 4, 2, 3, 7)$ and $H \cong K_4(1, 2, 6, 4, 4, 4)$. Hence

$$K_4(1, 4, 4, 2, 3, 7) \sim K_4(1, 2, 6, 4, 4, 4).$$

Case 2.3.2 $d' = f' = 4$. So $f = 7$. By Equation (2), $e + 1 = e'$. After simplification, we have $e = 3$ and $e' = 4$. We obtain the same solution as in Case 2.3.1.

Case 2.3.3 $e' = f' = 4$. So $e = 3$. By Equation (2), $f = d' + 3$. After simplification, we have $f = 7$ and $d' = 4$. We obtain the same solution as in Case 2.3.1.

Case 2.3.4 $d' = e' + 1 = 4$. So $e' = 3$ and $f = 7$. By Equation (2), $e + 2 = f'$. After simplification, we have $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

Case 2.3.5 $d' = f' + 1 = 4$. So $f' = 3$ and $f = 7$. By Equation (2), $e + 2 = e'$. After simplification, we have $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

Case B $e = 2$. So $d \geq 3$ and $f \geq 6$. We obtain the following after simplification.

$$Q_{15}(G) = -2s^4 - 2s^5 - s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5}.$$

$Q_{15}(H) = -s^6 - s^7 - s^{d'} - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{d'+8} + s^{e'+3} + s^{e'+6} + s^{f'+2} + s^{f'+7}$.

Consider the l.r.p in $Q_{15}(G)$ and the l.r.p in $Q_{15}(H)$, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$ or $d' = e' + 1 = 4$ or $d' = f' + 1 = 4$.

<u>**Case 1**</u> $d' = e' = 4$. We obtain the following after simplification.

$Q_{16}(G) = -s^5 - s^d - s^f - s^{f+1} + s^6 + s^{d+8} + s^{f+4} + s^{f+5}$.

$Q_{16}(H) = -s^6 - s^7 - s^{f'} - s^{f'+1} + s^{10} + s^{12} + s^{f'+2} + s^{f'+7}$.

The h.r.p in $Q_{16}(H)$ is 12 or $f' + 7$.

<u>**Case 1.1**</u> $12 \geq f' + 7$. The h.r.p in $Q_{16}(G)$ is $f + 5$ or $d + 8$. So we have $f + 5 = 12$ or $d + 8 = 12$.

<u>**Case 1.1.1**</u> $f + 5 = 12$. So $f = 7$. We obtain $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

<u>**Case 1.1.2**</u> $d + 8 = 12$. So $d = 4$. We obtain $G \cong H$.

<u>**Case 1.2**</u> $12 < f' + 7$. The h.r.p in $Q_{16}(G)$ is $f + 5$ or $d + 8$. So we have $f + 5 = f' + 7$ or $d + 8 = f' + 7$.

<u>**Case 1.2.1**</u> $f + 5 = f' + 7$. So $f = f' + 2$. By Equation (2), $d = 4$. Cancelling the equal terms in $Q_{16}(G)$ and $Q_{16}(H)$ gives the following.

$Q_{17}(G) = -s^5 - s^4 - s^f - s^{f+1} + s^6 + s^{f+4}$.

$Q_{17}(H) = -s^6 - s^7 - s^{f-2} - s^{f-1} + s^{10} + s^f$.

We obtain $f = 6$ and $f' = 4$. Therefore, $G \cong K_4(1,4,4,4,2,6)$ and $H \cong K_4(1,2,6,4,4,4)$. Thus, $G \cong H$.

Case 1.2.2 $d + 8 = f' + 7$. So $d + 1 = f'$. By Equation (2), $f = 7$. We obtain $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case 2 $d' = f' = 4$. So $e' \geq 4$. We obtain the following after simplification.

$$Q_{18}(G) = -s^5 - s^d - s^f - s^{f+1} + s^7 + s^{d+8} + s^{f+4} + s^{f+5}.$$

$$Q_{18}(H) = -s^6 - s^7 - s^{e'} - s^{e'+1} + s^{11} + s^{12} + s^{e'+3} + s^{e'+6}.$$

The h.r.p in $Q_{18}(H)$ is 12 when $e' = 4, 5$ or $e' + 6$ when $e' \geq 6$.

Case 2.1 $12 \geq e' + 6$. The h.r.p in $Q_{18}(G)$ is $f + 5$ or $d + 8$.

Case 2.1.1 $f + 5 = 12$ and $e' = 4$. So $f = 7$. By Equation (2), $d = 3$. We obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

Case 2.1.2 $d + 8 = 12$ and $e' = 4$. So $d = 4$. By Equation (2), $f = 6$. We obtain $G \cong H$.

Case 2.1.3 $f + 5 = 12$ and $e' = 5$. So $f = 7$. By Equation (2), $d = 4$. We obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

Case 2.1.4 $d + 8 = 12$ and $e' = 5$. So $d = 4$. By Equation (2), $f = 7$. We obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

Case 2.2 $12 < e' + 6$. The h.r.p in $Q_{18}(G)$ is $f + 5$ or $d + 8$. So we have $f + 5 = e' + 6$ or $d + 8 = e' + 6$.

Case 2.2.1 $f + 5 = e' + 6$. So $f = e' + 1$. By Equation (2), $d = 5$. We obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

Case 2.2.2 $d + 8 = e' + 6$. So $d + 2 = e'$. By Equation (2), $f = 8$. We obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

Case 3 $e' = f' = 4$. So $d' \geq 3$. We obtain the following after simplification.

$$Q_{19}(G) = -s^d - s^f - s^{f+1} + s^{d+8} + s^{f+4} + s^{f+5}.$$

$$Q_{19}(H) = -s^6 - s^7 - s^{d'} + s^{10} + s^{11} + s^{d'+8}.$$

Comparing the h.r.p in $Q_{19}(G)$ an the h.r.p in $Q_{19}(H)$, we have $f+5 = d'+8$ or $d + 8 = d' + 8$.

Case 3.1 $f + 5 = d' + 8$. So $f = d' + 3$. By Equation (2), $d = 3$. We obtain the following after simplification.

$$Q_{20}(G) = -s^3 - s^f - s^{f+1} + s^{f+4},\ Q_{20}(H) = -s^6 - s^7 - s^{f-3} + s^{10}.$$

So $f = 6$ and $d' = 3$. Therefore $G \cong K_4(1, 4, 4, 3, 2, 6)$ and $H \cong K_4(1, 2, 6, 3, 4, 4)$. Hence, $G \cong H$.

Case 3.2 $d + 8 = d' + 8$. So $d = d'$. By Equation (2), $f = 6$. We obtain $G \cong K_4(1, 4, 4, d, 2, 6)$ and $H \cong K_4(1, 2, 6, d, 4, 4)$. Hence, $G \cong H$.

Case 4 $d' = e' + 1 = 4$. So $e' = 3$. We obtain the following after simplification.

$$Q_{21}(G) = -2s^5 - s^d - s^f - s^{f+1} + s^7 + s^{d+8} + s^{f+4} + s^{f+5}.$$

$$Q_{21}(H) = -s^3 - s^6 - s^7 - s^{f'} - s^{f'+1} + s^9 + s^{12} + s^{f'+2} + s^{f'+7}.$$

Note that there are no positive terms in $Q_{21}(H)$ can be cancelled with the term $-2s^5$ in $Q_{21}(G)$ since $d \geq 3$ and $f \geq 6$. Thus a contradiction.

Case 5 $d' = f' + 1 = 4$. So $f' = 3$. We obtain the following after simplification.

$$Q_{21}(G) = -2s^5 - s^d - s^f - s^{f+1} + s^6 + s^7 + s^{d+8} + s^{f+4} + s^{f+5}.$$

$$Q_{21}(H) = -s^3 - s^6 - s^7 - s^{e'} - s^{e'+1} + s^5 + s^{10} + s^{12} + s^{e'+3} + s^{e'+6}.$$

Similar to Case 4 above, we obtain a contradiction.

Thus, from Subcases 1.1.1 of Case A, 2.3.1, 2.3.2 and 2.3.3 of Case B, we obtain the following result

$$K_4(1,4,4,2,3,7) \sim K_4(1,2,6,4,4,4).$$

This completes the proof. □

By Lemma 2.5 or using similar method to that of Lemmas 3.1 and 3.2, we can obtain Lemmas 3.3, 3.4 and 3.5.

Lemma 3.3. *If G is of type of $K_4(1,4,4,d,e,f)$ and H is of type of $K_4(2,2,5,d',e',f')$, then there is no graph satisfying $G \sim H$.*

Lemma 3.4. *If G is of type of $K_4(1,4,4,d,e,f)$ and H is of type of $K_4(1,2,c',2,e',4)$, then there is no graph satisfying $G \sim H$.*

Lemma 3.5. *If G is of type of $K_4(1,4,4,d,e,f)$ and H is of type of $K_4(1,2,c',4,e',2)$, then there is no graph satisfying $G \sim H$.*

Similarly, we can also prove the following lemmas.

Lemma 3.6. *If G is of type of $K_4(1,4,4,d,e,f)$ and H is of type of $K_4(1,3,c',2,e',3)$, then there is no graph satisfying $G \sim H$.*

Lemma 3.7. *If G is of type of $K_4(1,4,4,d,e,f)$ and H is of type of $K_4(1,2,c',3,e',3)$, then there is no graph satisfying $G \sim H$.*

Now we give the main result of the paper.

Theorem 3.1. *K_4-homeomorphs $K_4(1,4,4,d,e,f)$ with girth 9 is not χ-unique if and only if it is isomorphic to $K_4(1,4,4,4,2,6)$, $K_4(1,4,4,6,2,6)$, $K_4(1,4,4,2,3,7)$, $K_4(1,4,4,6,3,7)$, $K_4(1,4,4,6,3,8)$, , $K_4(1,4,4,3,5,8)$, $K_4(1,4,4,i,i+1,i+5)$ or $K_4(1,4,4,i+2,i,i+4)$, where $i \geq 3$.*

Proof. Let G and H be two graphs such that $G \cong K_4(1,4,4,d,e,f)$ and $H \sim G$. Since the girth of G is 9, there is at most one 1 among d,e,f. Moreover by Lemma 2.1 (ii) and (iii), it follows that H is a K_4-homoemorph with girth 9. So H must be one of the following 10 types.

Type 1: $K_4(1,2,6,d',e',f')$, *where* $d'+e' \geq 7, d'+f' \geq 3, e'+f' \geq 8$;

Type 2: $K_4(1,3,5,d',e',f')$, *where* $d'+e' \geq 6, d'+f' \geq 4, e'+f' \geq 8$;

Type 3: $K_4(1,4,4,d',e',f')$, *where* $d'+e' \geq 5, d'+f' \geq 5, e'+f' \geq 8$;

Type 4: $K_4(2,2,5,d',e',f')$, *where* $d'+e' \geq 7, d'+f' \geq 4, e'+f' \geq 7$;

Type 5: $K_4(2,3,4,d',e',f')$, *where* $d'+e' \geq 6, d'+f' \geq 5, e'+f' \geq 7$;

Type 6: $K_4(1,2,c',2,e',4)$, *where* $c' \geq 6, e' \geq 5$;

Type 7: $K_4(1,2,c',4,e',2)$, *where* $c' = e' \geq 6$;

Type 8: $K_4(1,2,c',3,e',3)$, *where* $c' \geq 6, e' \geq 5$;

Type 9: $K_4(1,3,c',2,e',3)$, *where* $c' = e' \geq 5$;

Type 10: $K_4(2, 2, c', 2, e', 3),\ where\ c' = e' \geq 5.$

From Lemmas 2.2–2.5, 3.1–3.7, we obtain the result as desired. This completes the proof of Theorem 3.1. □

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