Malaysian Journal of Mathematical Sciences 9(3): 409–416 (2015)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

# On Closed and Supersaturated Semigroups

Noor Alam *[1] and Noor Mohammad Khan [2]

[1] *College of Preparatory Year, University of Hail, K.S.A.*
[2] *Aligarh Muslim University, India*

*E-mail:*
* *alam.nooramu@gmail.com*

## ABSTRACT

After showing that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$, we have shown that every globally idempotent ideal of a supersaturated semigroup satisfying the identity $ax = axa[xa = axa]$ is supersaturated.

**Keywords:** Epimorphism, dominion, left[right] seminormal band, closed semigroup, supersaturated semigroup, zigzag equations.

# 1. Introduction

In Alam and Khan (2013), have shown that the class of all left[right] quasinormal bands is closed. In this paper, we have generalized this result to the class of all left[right] seminormal bands by zigzag manipulations and have shown that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$. As a corollary of this fact, we get that the class of all left[right] seminormal bands is closed.

In Higgins (1985), had shown that a commutative globally idempotent ideal $U$ of a supersaturated semigroup was supersaturated. In Khan and Shah (2010) generalized this result by taking $U$ as a permutative globally idempotent ideal satisfying a permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$ for which $i_1 = 1$ and $i_n \neq n$ and thus, relaxed the commutativity of $U$. In [2], Alam and Khan further generalized this result by taking $U$ as a permutative globally idempotent ideal satisfying a nontrivial permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{in}$ for which $i_1 = 1$ and $i_n = n$ and thus, relaxed the right semicommutativity of $U$.

We, now, further extend the above result by showing that a globally idempotent ideal of a supersaturated semigroup satisfying the identity $ax = axa[xa = axa]$ is supersaturated, and, thus, enlarge the class of supersaturated globally idempotent ideals of a supersaturated semigroup.

# 2. Preliminaries

Let $U$ and $S$ be any semigroups with $U$ a subsemigroup of $S$. Following Isbell (1966), we say that $U$ dominates an element $d$ of $S$ if for every semigroup $T$ and for all homomorphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all elements of $S$ dominated by $U$ is called the *dominion* of $U$ in $S$, and we denote it by $Dom(U, S)$. It may be easily seen that $Dom(U, S)$ is a subsemigroup of $S$ containing $U$. Let $\mathcal{C}$ be a class of semigroups. A semigroup $U$ is said to be $\mathcal{C}$-*closed* if for all $S \in \mathcal{C}$ such that $U$ is a subsemigroup of $S$, $Dom(U, S) = U$. Let $\mathcal{B}$ and $\mathcal{C}$ be classes of semigroups such that $\mathcal{B} \subseteq \mathcal{C}$. Then $\mathcal{B}$ is said to be $\mathcal{C}$-*closed* if every member of $\mathcal{B}$ is $\mathcal{C}$-*closed*. A class $\mathcal{C}$ of semigroups is said to be closed if for all $U, S \in \mathcal{C}$ with $U$ a subsemigroup of $S$, $Dom(U, S) = U$. If $Dom(U, S) \neq S$ for every properly containing semigroup $S$, then $U$ is said to be *saturated*. Following Higgins (1985), a semigroup $U$ is said to be *supersaturated* if every morphic image of $U$ is saturated.

A morphism $\alpha : S \to T$ is said to be an *epimorphism* (epi for short) if for all morphisms $\beta, \gamma$; $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$ (where $\beta, \gamma$ are semigroup morphisms).

It can be easily checked that $\alpha : S \to T$ is epi if and only if $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$ . Onto morphisms are always epimorphisms, but the converse is not true in general in the category of all semigroups. Infact every epimorphism from a semigroup $U$ is onto is just to say that $U$ is supersaturated.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

**Result 2.1** (Isbell (1966), Theorem 2.3 or Howie (1976), Theorem 2.13). Let $U$ be a subsemigroup of a semigroup $S$ and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of $d$ as follows:

$$d = a_0 y_1 = x_1 a_1 y_1 = x_1 a_2 y_2 = x_2 a_3 y_2 = \cdots = x_m a_{2m-1} y_m = x_m a_{2m}, \tag{2.1}$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $x_i, y_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = x_1 a_1, \qquad a_{2m-1} y_m = a_{2m},$$
$$a_{2i-1} y_i = a_{2i} y_{i+1}, \qquad x_i a_{2i} = x_{i+1} a_{2i+1} \qquad (1 \leq i \leq m - 1).$$

Such a series of factorization is called a *zigzag* in $S$ over $U$ with value $d$, length $m$ and spine $a_0, a_1, \ldots, a_{2m}$.

We refer to the equations in Result 2.1, in whatever follows, as *the zigzag equations.*

Recall that a *band* $B$ (a semigroup in which every element is an idempotent) is *left* [*right*] *regular* if it satisfies the identity $ax = axa[xa = axa]$; *left[right] quasi normal* if it satisfies the identity $axy = axay[yxa = yaxa]$ and *left*[*right*] *seminormal* if it satisfies the identity $axy = axyay[yxa = yayxa]$, (see Petrich and Reilly (1999)). Clearly the class of all left[right] seminormal bands contains the class of all left[right] quasinormal bands as well as the class of all left[right] regular bands. A semigroup $S$ is said to be *globally idempotent* if for all $s \in S$, there exist $x, y \in S$ such that $s = xy$ or equivalently $S^2 = S$, (see Petrich (1973)).

For any unexplained symbols and terminology, reader may refer to Clifford and Preston (1967) and Howie (1976). Also, in whatever follows, bracket has been used for the dual notion [statement] to the other notion [statement].

# 3. Main Results

## 3.1 On Closed Semigroups

**Theorem 3.1.** *Let $\mathcal{B}$ be the class of all left seminormal bands and $\mathcal{C}$ be the class of all semigroups satisfying the identity $axy = axyay$. Then $\mathcal{B}$ is $\mathcal{C}$-closed.*

*Proof.* Let $U$ and $S$ be a left seminormal band and a semigroup satisfying the identity $axy = axyay$ respectively with $U$ a subsemigroup of $S$. Take any $d$ in $Dom(U, S) \setminus U$. Then, by Result 2.1, we may let (2.1) be a zigzag in $S$ over $U$ with value $d$ of minimal length $m$.

Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations as } a_0 = x_1a_1) \\
&= (x_1a_1a_2)y_2 && \text{(by zigzag equations as } a_1 \text{ is an idempotent and } a_1y_1 = a_2y_2) \\
&= (x_1a_1a_2x_1a_2)y_2 && \text{(by definition of } S) \\
&= x_1a_1a_2x_2a_3y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= x_1a_1a_2x_2a_3a_3y_2 && \text{(as } a_3 \text{ is an idempotent)} \\
&= (x_1a_1a_2x_1a_2)a_3y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= (x_1a_1a_2)a_3y_2 && \text{(by definition of } S) \\
&= a_0a_2(a_3y_2) && \text{(by zigzag equations)} \\
&= (\prod_{i=0}^{1} a_{2i})(a_3y_2) \\
&\vdots \\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}y_{m-1}) \\
&= (x_1a_1a_2)a_4 \cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations as } a_{2m-3}y_{m-1} = a_{2m-2}y_m) \\
&= (x_1a_1a_2x_1a_2)a_4 \cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by definition of } S)
\end{aligned}$$

$$
\begin{aligned}
&= (x_1a_1a_2x_2a_3)a_4\cdots a_{2m-4}(a_{2m-2}y_m) \quad \text{(by zigzag equations as } x_1a_2 = x_2a_3\text{)}\\
&= x_1a_1a_2(x_2a_3a_4)\cdots a_{2m-4}(a_{2m-2}y_m)\\
&= x_1a_1a_2(x_2a_3a_4x_2a_4)\cdots a_{2m-4}(a_{2m-2}y_m) \quad \text{(by definition of } S\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots a_{2m-4}(a_{2m-2}y_m) \quad \text{(by zigzag equations as } x_2a_4 = x_3a_5\text{)}\\
&\vdots\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-2}a_{2m-4}(a_{2m-2}y_m)\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2})y_m\\
&\qquad \text{(by zigzag equations as } x_{m-2}a_{2m-4} = x_{m-1}a_{2m-3}\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2}x_{m-1}a_{2m-2})y_m \quad \text{(by definition of } S\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-1}a_{2m-3}a_{2m-2}x_ma_{2m-1}y_m\\
&\qquad \text{(by zigzag equations as } x_{m-1}a_{2m-2} = x_ma_{2m-1}\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2}x_{m-1}a_{2m-2})a_{2m-1}y_m\\
&\qquad \text{(by zigzag equations as } a_{2m-1} \text{ is an idempotent and } x_{m-1}a_{2m-2} = x_ma_{2m-1}\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2})a_{2m-1}y_m \quad \text{(by definition of } S\text{)}\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m-1}y_m\\
&\qquad \text{(by zigzag equations as } x_{m-2}a_{2m-4} = x_{m-1}a_{2m-3}\text{)}\\
&\vdots\\
&= (x_1a_1a_2x_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m\\
&= (x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \quad \text{(by definition of } S\text{)}\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} \quad \text{(by zigzag equations)}\\
&= (\prod_{i=0}^{m} a_{2i}) \in U.
\end{aligned}
$$

$\Rightarrow d \in U.$

Hence, $Dom(U, S) = U$ . □

Dually we may prove the following :

**Theorem 3.2.** *Let $\mathcal{B}$ be the class of all right seminormal bands and $\mathcal{C}$ be the class of all semigroups satisfying the identity $yxa = yayxa$. Then $\mathcal{B}$ is $\mathcal{C}$-closed.*

**Corollary 3.1.** *The class of all left[right] seminormal bands is closed.*

## 3.2 On Supersaturated Semigroups

A semigroup $S$ is said to be right reductive with respect to $X$ if $xa = xb$ for all $x$ in $X \Rightarrow a = b$ $(a, b \in S)$, where $X$ is a subset of $S$. A semigroup $S$ is said to be supersaturated if every homomorphic image of $S$ is saturated. The following result is from Higgins (1984).

**Result 3.2.1** (Higgins (1984), Theorem 8). A semigroup $U$ is saturated [supersaturated] if the ideal $U^n$ is saturated [supersaturated] (for some natural number $n$). In particular a finite semigroup is saturated [supersaturated] if the ideal generated by the idempotents is saturated [supersaturated].

It is still an open question whether or not the converse of the above result is true. Since $\Im_X$, the semigroup of all transformations on the set $X$, is absolutely closed, subsemigroups of absolutely closed or saturated semigroups need not be absolutely closed or saturated in general. However it is not known whether or not an ideal of a saturated [absolutely closed] semigroup is saturated [absolutely closed]. In Higgins (1985), had shown that the converse of the above result held in some special cases and proved that if $S$ was supersaturated commutative semigroup, then every commutative globally idempotent ideal of $S$ was supersaturated. He, in fact, had proven the following:

**Result 3.2.2** (Higgins (1985), Theorem 14). Let $S$ be a supersaturated semigroup and suppose that $U$ is a commutative ideal of $S$ such that $U^n$ is globally idempotent for some natural number n. Then $U$ is supersaturated.

The following result is very crucial for the proof of our next theorem which extends the class of supersaturated globally idempotent ideals of a supersaturated semigroup.

**Result 3.2.3** (Khan and Shah (2010), Lemma 2.5). Suppose that a globally idempotent semigroup $U$ is not supersaturated. Then there exists a non-surjective epimorphism $\phi : U \to V$ such that $V$ is right reductive with respect to $U\phi$.

**Theorem 3.2.4** Let $S$ be a supersaturated semigroup and let $U$ be any ideal of $S$ satisfying the identity $axa = ax$. If $U^n$ is globally idempotent for some natural number $n$, then $U$ is supersaturated.

*Proof.* By Result 3.2.1, the theorem is proved if we prove that $U^n$ is supersaturated. Therefore, without loss of generality, we may assume that $U$ be a globally idempotent ideal satisfying the identity $axa = ax$. Suppose to contrary that $U$ were not supersaturated. Then, by Result 3.2.3, there exists a non-surjective epimorphism $\phi : U \to \overline{V}$ such that $\overline{V}$ is right reductive with respect to $U\phi$, denoted by $\overline{U}$ (up to isomorphism). Let $\rho = \phi o \phi^{-1} \cup 1_s$. It is clear that $\rho$ is an equivalence relation on $S$. Next we show that $\rho$ is a congruence on $S$. For this, we require to show that if $u, v \in U$ and $w \in S \backslash U$, then $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$ and $(wu)\phi = (wv)\phi$. We will prove only the first equality as rest of the proof including that of the latter equality follows exactly on the same lines as in the proof of Khan and Shah (2010) (Theorem 2.6).

Suppose that $u, v \in U$, $w \in S \backslash U$ and $(uw)\phi \neq (vw)\phi$. Since $\overline{V}$ is right reductive with respect to $\overline{U}$, there exists $x \in U$ such that $x\phi(uw)\phi \neq x\phi(vw)\phi$. Then $(x(uw))\phi \neq (x(vw))\phi$ which implies that $((xu)w)\phi \neq ((xv)w)\phi$. Since $U$ satisfies identity $axa = ax$. Therefore we have $((xux)w)\phi \neq ((xvx)w)\phi \Rightarrow ((xu)xw)\phi \neq ((xv)xw)\phi \Rightarrow (xu)\phi(xw)\phi \neq (xv)\phi(xw)\phi$. which in turn implies that $(xu)\phi \neq (xv)\phi$ again, it implies that $x\phi u\phi \neq x\phi v\phi$ which in turn implies that $u\phi \neq v\phi$. Therefore the statement $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$. Hence $\rho$ is a right congruence. □

Dually, we may prove the following:

**Theorem 3.3.** *Let $S$ be a supersaturated semigroup and let $U$ be any ideal of $S$ satisfying the identity $axa = xa$. If $U^n$ is globally idempotent for some natural number $n$, then $U$ is supersaturated.*

# 4. Conclusion

In this paper, by using zigzag manipulations, we have proved that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$ which is generalization of the result that the class of all left[right] quasi-normal bands is closed. Finally, we have shown that every globally idempotent ideal of a supersaturated semigroup satisfying the identity $ax = axa[xa = axa]$ is supersaturated.

# References

Alam, N. and Khan, N. (2013). Special semigroup amalgams of quasi unitary subsemigroups and of quasi normal bands. *Asian European Journal of Mathematics.*

Clifford, A. and Preston, G. (1961, 1967). The algebraic theory of semigroups. *Math. Surveys*, 1, 2.

Higgins, P. (1984). Saturated and closed varieties of semigroups. *J. Austral. Math. Soc.*, 36:153–175.

Higgins, P. (1985). Epimorphisms, dominions and semigroups. *Algebra Univers.*, 21:225–233.

Howie, J. (1976). *An Introduction to Semigroup Theory*, volume 7. Academic Press, San Diego. London Math. Soc. Monograph.

Isbell, J. (1966). Epimorphisms and dominions. pages 232–246. Proc. of the conference on categorical Algebra.

Khan, N. M. and Shah, A. H. (2010). Epimorphisms dominions and permutative semigroups. *Semigroup Forum*, 80:181–190.

Petrich, M. (1973). *Introduction to semigroups.* E. Merrill Publishing Co, Columbus, Ohio.

Petrich, M. and Reilly, N. (1999). *Completely Regular Semigroups.* John Wiley and Sons. Canadian Math. Soc. Monograph.