Malaysian Journal of Mathematical Sciences 9(3): 481-490 (2015)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Discrete Pursuit Game when Controls are subjected to Geometric and Total Constraints

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ABSTRACT

We study two linear discrete game problems, where dynamics of pursuer and evader are described by two different type equations. The terminal set is a subset of R^n. In the first problem, controls of players are subjected to geometric constraints, and, in the second one, controls are subjected to total constraints. We obtain sufficient conditions of completion pursuit from certain initial positions of the players in finite time interval. Strategy of the pursuer is constructed based on information about initial positions of players and current value of the control parameter of the evader.

Keywords: Game, pursuer, evader, terminal set, subset, controls of players, strategy, position.

1. Introduction

Several literatures have addressed differential and discrete games where control parameters are subjected to various constraints (see all references).

The purpose of the paper (Satimov and Ibragimov (2004)) was to establish conditions so that pursuit can be completed from any given initial position of players in a discrete game of many players. Such condition was obtained in terms of eigenvalues of matrices in the equation and resources of players.

Azamov and Kuchkarov (2010) deals with the problems of controllability and pursuit for linear discrete systems. The main result of this paper is necessary and sufficient condition of equivalence of solvability of the zero-controllability and pursuit problem when controls of the pursuer and evader are subjected to geometric and total constraints respectively.

In Ibragimov (2005), two problems of pursuit were investigated for linear discrete games. Both problems are considered under geometric constraints on control of the pursuer. As to the control of the evader, in the first game problem, it is subjected to geometric constraint, and in the second game problem, it satisfies total constraint. Such type of constraints is known as the analogue of the integral constraints. Necessary and sufficient conditions were obtained to complete pursuit from any initial positions of players.

Also, in the paperKuchkarov et al. (2013), a linear discrete game is considered under total constraints on control vectors. Some conditions were found to complete the game from all points of the space where game is considered. It is assumed that modulus of all eigenvalues of the matrix in the equation are less than 1. An important point of the main result of the paper is that control resource of the pursuer may be less than that of the evader.

In the present paper, we study a pursuit discrete game, where dynamics of the players are described by linear discrete equations of different type. We consider two problems. In Problem 1, we impose geometric constraints on controls of the players, and, in Problem 2, controls are subjected to integral constraints. We obtain conditions which are sufficient to complete pursuit from certain initial positions of the players in finite time interval.

2. Statement of the Problem

Motions of the pursuer and evader are described by the following discrete equations

$$x(k) = Ax(k-1) + U(k) \qquad (1)$$

$$y(k) = By(k-1) + V(k) \qquad (2)$$

where $x(k), y(k), U(k), V(k) \in R^n$, $n \geq 1$, $k = 1, 2, ...$, k is step number, A, B are $n \times n$ constant matrices, U and V are control parameters of the pursuer and evader, respectively.

We consider two game problems. In the first game problem, controls of the players are subjected to geometric constraints, and, in the second one, they are subjected to total constraints. Controls U and V of the pursuer and evader are defined as the sequences

$$U = U(\cdot) = \{U(1), U(2), ..., U(k), ...\},\ V = V(\cdot) = \{V(1), V(2), ..., V(k), ...\}.$$

In the first game problem, controls satisfy the following geometric constraints:

$$U(k) \in P, \qquad (3)$$

$$V(k) \in Q, \qquad (4)$$

where P, Q - convex compact sets of the space R^n and, in the second game problem, controls satisfy the following total constraints:

$$\sum_{k=1}^{\infty} |U(k)|^2 < \rho_1^2 \qquad (5)$$

$$\sum_{k=1}^{\infty} |V(k)|^2 < \sigma_1^2 \qquad (6)$$

Definition 2.1.A sequence

$$U = (U(1), U(2), ...),\ U(k) = U(x(k), y(k), V(k)),\ U(k): R^{3n} \to R^n,$$

where $U(k)$ for each k satisfies the inclusion (3) (respectively inequality (5)), is called strategy of the pursuer in the game (1)-(4) (respectively (1)-(2), (5)-(6)).

Definition 2.2. Given initial position $z_0 = (x_0, y_0)$, $Cy_0 - Dx_0 \notin M$. If for a strategy of the pursuer and any control of the evader the solutions of the equations (1), (2)

$$x = x(\cdot) = \{x(1), x(2), ..., x(k), ...\},\ y = y(\cdot) = \{y(1), y(2), ..., y(k), ...\}$$

satisfy the inclusion

$$Cy(k) - Dx(k) \in M, \qquad (7)$$

at some $k \le N = N(z_0)$, then we say that pursuit can be completed for N steps, where C and D are given $n \times n$ constant matrices, and M is a subset of $\mathbb{R}^n$.

The pursuer tries to realize the inclusion (7) as soon as possible, and the evader's purpose is opposite.

Problem 1. Find sufficient conditions of completion of pursuit in the game (1)-(4).

Problem 2. Find sufficient conditions of completion of pursuit in the game (1)-(2), (5)-(6).

3. Main Result

3.1. The case of geometric constraints.

We'll now formulate the main hypotheses in the form of assumption.

Assumption 3.1. There exist compact sets P_1, P_2, and linear mappings $F(k,i): Q \to P_1, 1 \le i \le k,$ such that $P_1 + P_2 \subset P$ and the following conditions hold:

(i) $CB^k y_0 - DA^k x_0 \in D(A^{k-1} + A^{k-2} + ... + A + E)P_2$,

(ii) $\sum_{i=1}^{k} (CB^{k-i} - DA^{k-i} F(k,i))Q \subset M$.

Theorem 3.1. Let there exist a step $N(z_0)$ for the point z_0 such that Assumption 1 holds true at $k = N(z_0)$. Then pursuit can be completed in the game (1)-(4) from the point z_0 for $N(z_0)$ steps.

Proof. Let $V = V(i)$ $, i = 1, 2, ..., k$ be an admissible control of the evader. Set

$$U(i) = F(k,i)V(i) + w(i), i = 1, 2, ..., k$$

where $w(i) \in P_2$, $i = 1, 2, ..., k$, is unknown vector which will be specified below. Then for the solutions of the equations (1), (2) with initial conditions $x(0) = x_0$, $y(0) = y_0$ we obtain on the step k

$$
\begin{aligned}
Cy(k)-Dx(k) &= CB^k y_0 + CB^{k-1}V(1) + CB^{k-2}V(2) + \ldots + CBV(k-1) \\
&+ CV(k) - DA^k x_0 \\
&- \left[DA^{k-1}F(k,1)V(1) + DA^{k-2}F(k,2)V(2) + \ldots \right] \\
&+ DAF(k,k-1)V(k-1) + DF(k,k)V(k) \\
&-(DA^{k-1}w(1) + DA^{k-2}w(2) + \ldots + DAw(k-1) + Dw(k)) \\
&= CB^k y_0 - DA^k x_0 - D(A^{k-1}w(1) + A^{k-2}w(2) + \ldots \\
&+ Aw(k-1) + w(k)) + \sum_{i=1}^{k} (CB^{k-i} - DA^{k-i}F(k,i))V(i). \quad (8)
\end{aligned}
$$

We consider the following equation at $N = N(z_0)$

$$
CB^N y_0 - DA^N x_0 = D(A^{N-1} + A^{N-2} + \ldots + A + E)w \quad (9)
$$

with respect to unknown vector $w \in P_2$. According to the condition (i) of Assumption 1, there exists a solution of the equation (9). Denote by $\bar{w}$ its lexicographically minimum solution. In (8), we let $w(i) = \bar{w}$, $1 \le i \le N$, and then using the condition (ii) of Assumption 1, we have at $k = N$ that

$$
\begin{aligned}
&Cy(N) - Dx(N) = CB^N y_0 - DA^N x_0 - D(A^{N-1} + A^{N-2} + \ldots + A + E)\bar{w} + \\
&+ \sum_{i=1}^{N} (CB^{N-i} - DA^{N-i}F(N,i))V(i) \in \sum_{i=1}^{N} (CB^{k-i} - DA^{k-i}F(N,i))Q \subset M.
\end{aligned}
$$

Admissibility of the control $U(k)$ follows from the admissibility of $V(k)$ and the choice of $\bar{w}$. Proof of Theorem 1 is complete.

Assumption 3.2. Let

(i) $\det(A) \cdot \det(D) \ne 0$, $P = \rho S$, $Q = \sigma S$, and there a number $d > 0$ exists such that for any $k \ge 1$ $\left\| (A^k)^{-1} D^{-1} CB^k \right\| \le d$, where A^{-1} is the inverse of the matrix A, $\|A\|$ is operator norm of the matrix A, S is unit ball of R^n centered at the origin.

(ii) $\rho > d\sigma$.

Theorem 3.2. Let Assumption 2 hold true. If for the point z_0 there exists a step $N(z_0)$ and vector $m_0 \in M$ such that

$$-m_0 + CB^k y_0 - DA^k x_0 \in (\rho - d\sigma)D(A^{k-1} + A^{k-2} + ... + A + E)S \qquad (10)$$

at $k = N(z_0)$, then pursuit can be completed from the point z_0 for $N(z_0)$ steps.

Proof. Let $V = V(i),\ i = 1,2,.....,N$, be any control of the evader. We suggest to the pursuer the following strategy

$$U(i) = (A^{N-i})^{-1} D^{-1} CB^{N-i}\ V(i) + w\ ,\ i = 1,2,.....,N, \qquad (11)$$

where the vector $w \in (\rho - d\sigma)S$ will be specified later.

Since $A^k (A^k)^{-1} = E$, $k = 1,2,...,N$, therefore for the solutions of the equations (1) and (2) with the initial conditions $x(0) = x_0$, $y(0) = y_0$ we obtain

$$Cy(N) - Dx(N) = C(B^N y_0 + B^{N-1}V(1) + B^{N-2}V(2) + ... + BV(N-1) + V(N)) - DA^N x_0$$

$$-D\left[D^{-1}CB^{N-1}V(1) + D^{-1}CB^{N-2}V(2) + ... + D^{-1}CBV(N-1) + D^{-1}CV(N)\right]$$
$$-D(A^{N-1} + A^{N-2} + ... + A + E)w$$
$$= CB^N y_0 - DA^N x_0 - D(A^{N-1} + A^{N-2} + ... + A + E)w. \qquad (12)$$

We consider the equation

$$-m_0 + CB^N y_0 - DA^N x_0 = D(A^{N-1} + A^{N-2} + ... + A + E)w \qquad (13)$$

with respect to unknown vector $w \in (\rho - d\sigma)S.$. It follows from the condition (ii) of Assumption 3.2 and inclusion (10) that the equation (13) has a solution. Denote it by $\breve{w}$. Letting $w = \breve{w}$ in (12) we see that

$$Cy(N) - Dx(N) = -m_0 + CB^N y_0 - DA^N x_0 - D(A^{N-1} + A^{N-2} + ... + A + E)\breve{w} + m_0$$

$$= m_0 \in M.$$

This proves that pursuit can be completed in the game (1), (2). It remains to show admissibility of the strategy (11). Indeed, we use conditions (i) and (ii) of Assumption 3.2 to obtain

$$|U(i)| \le |(A^{(N-i)})^{-1} D^{-1} C B^{N-i} V(i)| + |\breve{w}| \le \|(A^{(N-i)})^{-1} D^{-1} C B^{N-i}\| |V(i)| + |\breve{w}|$$
$$\le d\sigma + \rho - d\sigma = \rho$$

for all $i \ge 1$ and the proof of Theorem 3.2 is complete.

Assumption 3.3. Let (i) $B = A$; (ii) $\rho > \sigma$; (iii) there exists a vector $m_1 \in R^n$ such that $(m_1 + (C - D)\sum_{i=1}^{k} A^{k-i} Q \subset M$.

Theorem 3.3. Let Assumption 3.3 hold true. If for the point z_0 there exists a step $N(z_0)$ such that

$$-m_1 + CB^k y_0 - DA^k x_0 \in (\rho - \sigma) D (A^{k-1} + A^{k-2} + ... + A + E) S$$

at $k = N(z_0)$, then pursuit can be completed from the point z_0 for $N(z_0)$ steps.

Proof of Theorem 3.3 is similar to that of Theorem 3.1.

3.2. The case of total constraints.

We turn now to the game (1), (2), (5), (6).

Assumption 3.4. There exist linear mappings $F(k,i): R^n \to R^n, 1 \le i \le k,$ and vector $m_2 \in R^n$ such that

(i) $\|F(k,i)\| \le d_1, 1 \le i \le k$, $\rho_1 > d_1 \sigma_1$;

(ii) $-m_2 + CB^k y_0 - DA^k x_0 \in \frac{1}{\sqrt{k}} (\rho_1 - d_1 \sigma_1) D (A^{k-1} + A^{k-2} + ... + A + E) S;$

(iii) $m_2 + G \subset M$, where

$$G = \left\{ \sum_{i=1}^{k} (CB^{k-i} - DA^{k-i} F(k,i)) V(i) \,\middle|\, \sum_{i=1}^{k} |V(i)|^2 \le \sigma_1^2 \right\};$$

Theorem 3.4. Let there exist a step $N(z_0)$ for the point z_0 to satisfy Assumption 3.4 at $k = N(z_0)$. Then pursuit can be completed in the game (1), (2), (5), (6) from the initial point z_0 for $N(z_0)$ steps.

Proof. Let $V = V(i)$, $i = 1, 2, ..., k$,be an arbitrary control of the evader. The strategy of the pursuer is chosen as follows

$$U(i) = F(k,i)V(i) + w, \; i = 1, 2, ..., k$$

where w is a vector from the set $\frac{1}{\sqrt{N}}(\rho_1 - d_1\sigma_1)S$ to be specified. It follows from (1) and (2) that

$$\begin{aligned} Cy(k) - Dx(k) &= CB^k y_0 - DA^k x_0 \\ &+(CB^{k-1} - DA^{k-1}F(k,1))V(1) + (CB^{k-2} - DA^{k-2}F(k,2))V(2) + ... \\ &+(CB - DAF(k,k-1))V(k-1)\ (C - DF(k,k))V(k) \\ &-D(A^{k-1} + A^{k-2} + ... + A + E)w, \end{aligned} \tag{14}$$

By condition (ii) of Assumption 3.4 the equation

$$-m_2 + CB^N y_0 - DA^N x_0 = D(A^{N-1} + A^{N-2} + ... + A + E)w \tag{15}$$

has a solution $w = \hat{w} \in \frac{1}{\sqrt{N}}(\rho_1 - d_1\sigma_1)S$. In (14), we set $w = \hat{w}$. Then in view of the condition (iii) of Assumption 3.4 we see at $k = N$ that

$$Cy(N) - Dx(N) = m_2 + \sum_{i=1}^{N}(CB^{N-i} - DA^{N-i}F(N,i))V(i) \in m_2 + G \subset M$$

which proves that pursuit can be completed. It remains to prove that

$$U(k) = F(N,k)V(k) + \hat{w}, \; k = 1, 2, ..., N,$$

satisfies the constraint (14). Indeed, since $V = V(k)$, $k = 1, 2, ..., N,$ satisfies the condition (4), and $\hat{w} \in \frac{1}{\sqrt{N}}(\rho_1 - d_1\sigma_1)S$, using the Minkowski inequality yields

$$\sum_{k=1}^{N}|U(k)|^2=\sum_{k=1}^{N}\left|F(N,k)V(k)+\widehat{w}\right|^2\le\left[\left(d_1^2\sum_{k=1}^{N}|V(k)|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{N}|\widehat{w}|^2\right)^{\frac{1}{2}}\right]^2$$

$$\le\left[d_1\sigma_1+\frac{1}{N}(\rho_1-d_1\sigma_1)N\right]^2=\rho_1^2.$$

Proof of Theorem 3.4 is complete.

4. Conclusion

We have obtained sufficient conditions of completion of pursuit in linear discrete systems under total constraints as well as geometric constraints on controls of the players. These conditions are new and can be applied for further investigation of discrete games. It should be noted that control of the pursuer $u(k)$ is constructed based on information about initial position and the value of control parameter of the evader, $v(k)$.

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