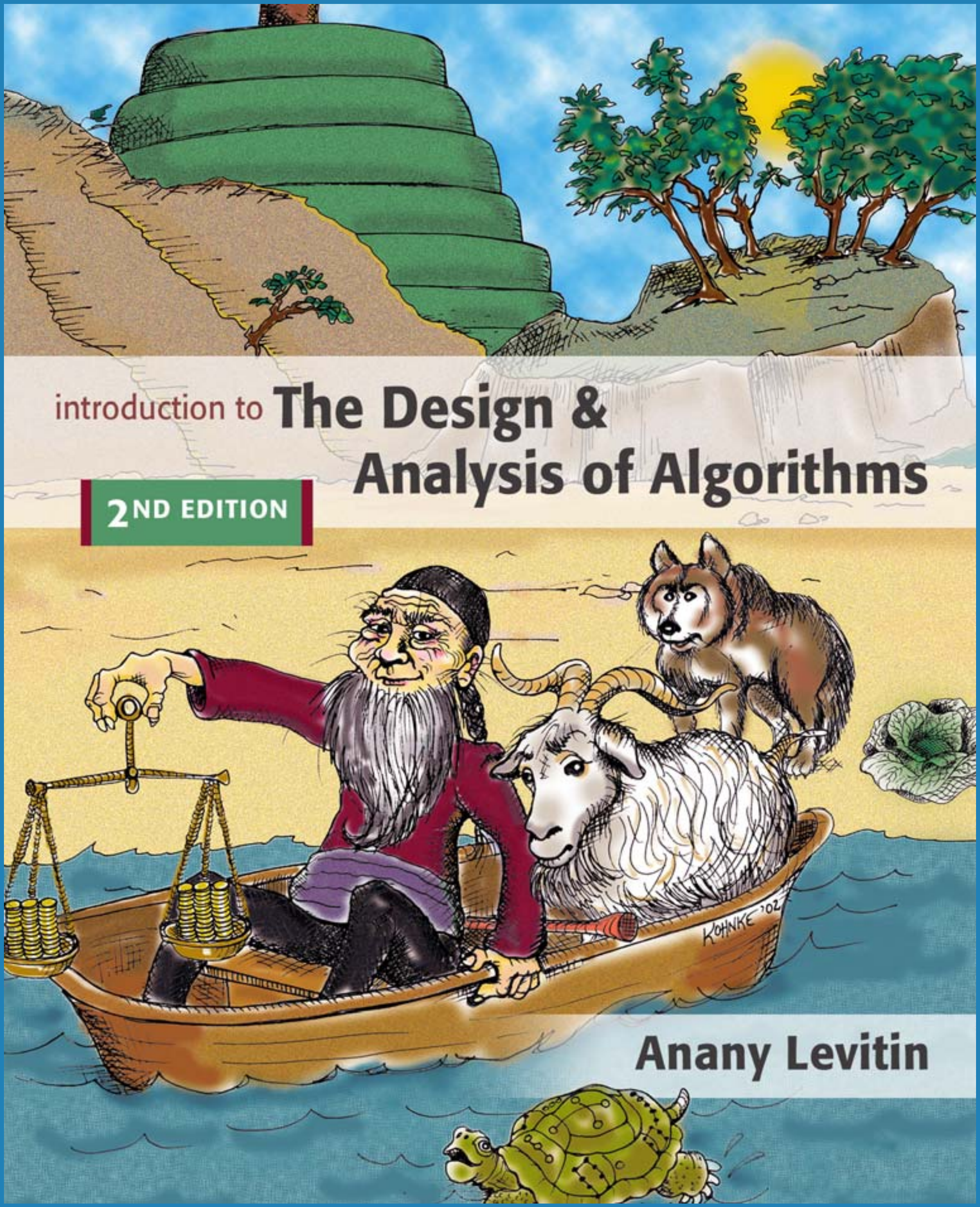
Chapter 2

Fundamentals of the Analysis of Algorithm Efficiency



Analysis of algorithms

- **Issues:**
  - **correctness**
  - **time efficiency**
  - **space efficiency**
  - **optimality**

- **Approaches:**
  - **theoretical analysis**
  - **empirical analysis**

Theoretical analysis of time efficiency

Time efficiency is analyzed by determining the number of repetitions of the *basic operation* as a function of *input size*

- ***Basic operation*: the operation that contributes most towards the running time of the algorithm**

input size

$$T(n) \approx c_{op} C(n)$$

running time

execution time for basic operation

Number of times basic operation is executed

Input size and basic operation examples

Problem	*Input size measure*	*Basic operation*
Searching for key in a list of *n* items	**Number of list's items, i.e. *n***	**Key comparison**
Multiplication of two matrices	**Matrix dimensions or total number of elements**	**Multiplication of two numbers**
Checking primality of a given integer *n*	***n'*size = number of digits (in binary representation)**	**Division**
Typical graph problem	**#vertices and/or edges**	**Visiting a vertex or traversing an edge**

Empirical analysis of time efficiency

- Select a specific (typical) sample of inputs

- Use physical unit of time (e.g., milliseconds)

  or

  Count actual number of basic operation's executions

- Analyze the empirical data

Best-case, average-case, worst-case

For some algorithms efficiency depends on form of input:

- Worst case: $C_{worst}(n)$ – maximum over inputs of size n
- Best case: $C_{best}(n)$ – minimum over inputs of size n
- Average case: $C_{avg}(n)$ – "average" over inputs of size n
  - Number of times the basic operation will be executed on typical input
  - NOT the average of worst and best case
  - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs

Example: Sequential search

ALGORITHM *SequentialSearch*$(A[0..n-1], K)$

```
//Searches for a given value in a given array by sequential search
//Input: An array A[0..n − 1] and a search key K
//Output: The index of the first element of A that matches K
//            or −1 if there are no matching elements
i ← 0
while i < n and A[i] ≠ K do
    i ← i + 1
if i < n return i
else return −1
```

- **Worst case**

- **Best case**

- **Average case**

Types of formulas for basic operation's count

- **Exact formula**

  e.g., $C(n) = n(n-1)/2$

- **Formula indicating order of growth with specific multiplicative constant**

  e.g., $C(n) \approx 0.5\, n^2$

- **Formula indicating order of growth with unknown multiplicative constant**

  e.g., $C(n) \approx cn^2$

Order of growth

- **Most important: Order of growth within a constant multiple as $n \to \infty$**

- **Example:**
  - **How much faster will algorithm run on computer that is twice as fast?**
  - **How much longer does it take to solve problem of double input size?**

Values of some important functions as $n \to \infty$

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Table 2.1 Values (some approximate) of several functions important for analysis of algorithms

Asymptotic order of growth

A way of comparing functions that ignores constant factors and small input sizes

- O($g(n)$): class of functions $f(n)$ that grow no faster than $g(n)$

- Θ($g(n)$): class of functions $f(n)$ that grow at same rate as $g(n)$

- Ω($g(n)$): class of functions $f(n)$ that grow at least as fast as $g(n)$

Big-oh

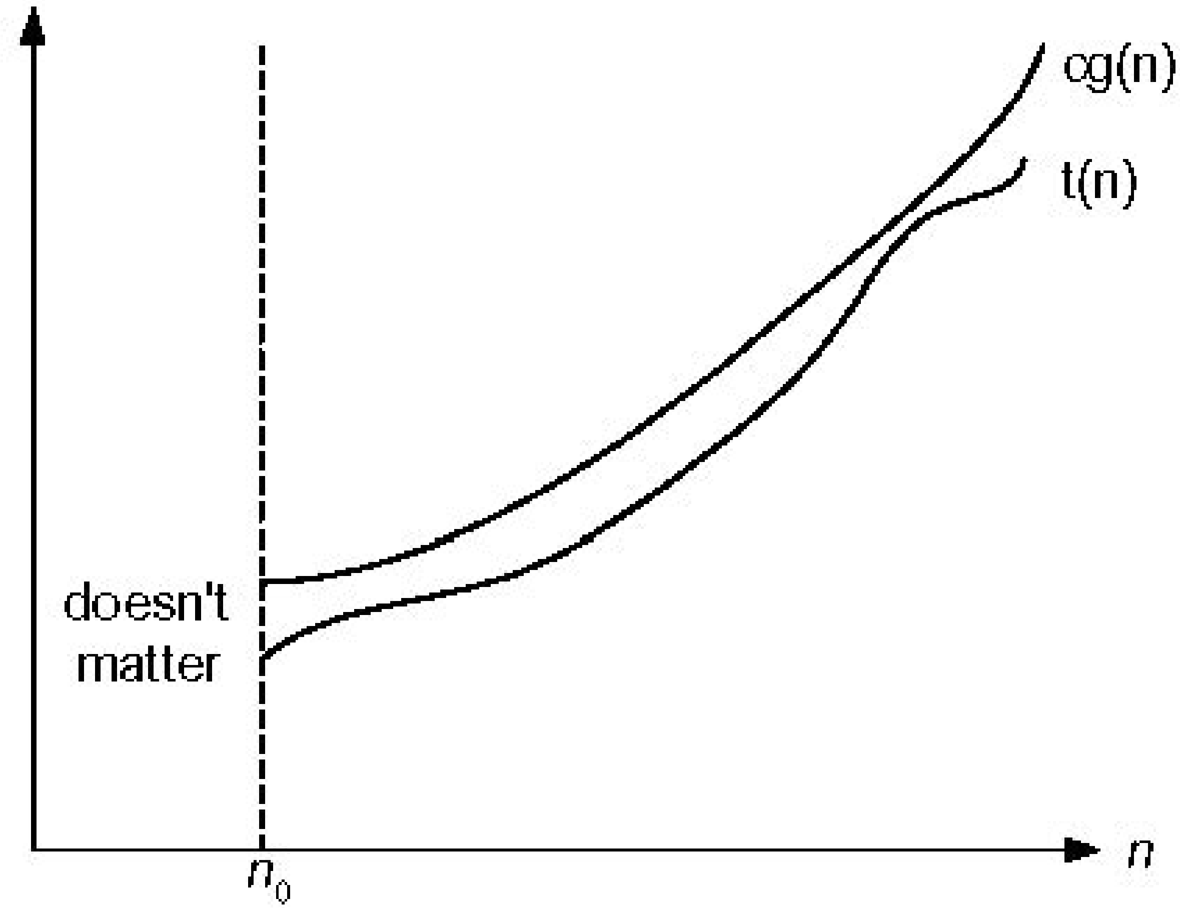


Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

Big-omega

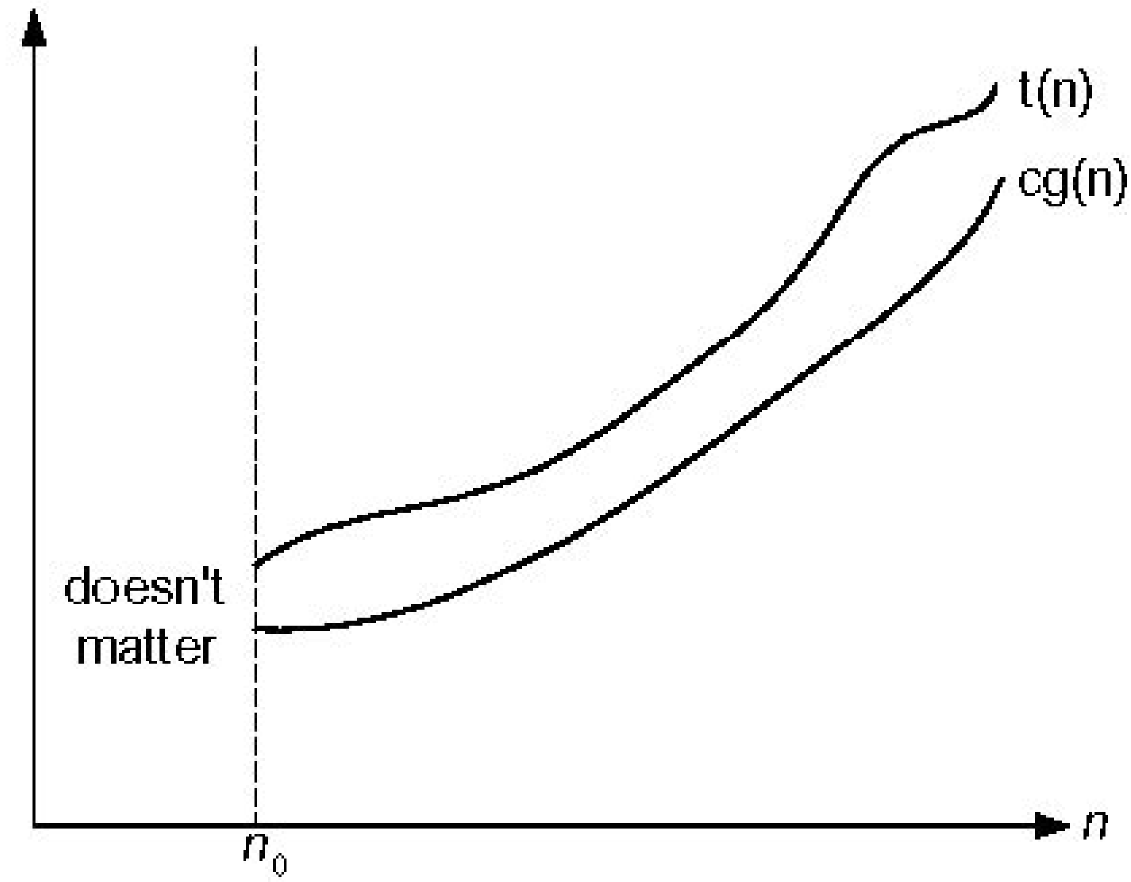


Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

Big-theta

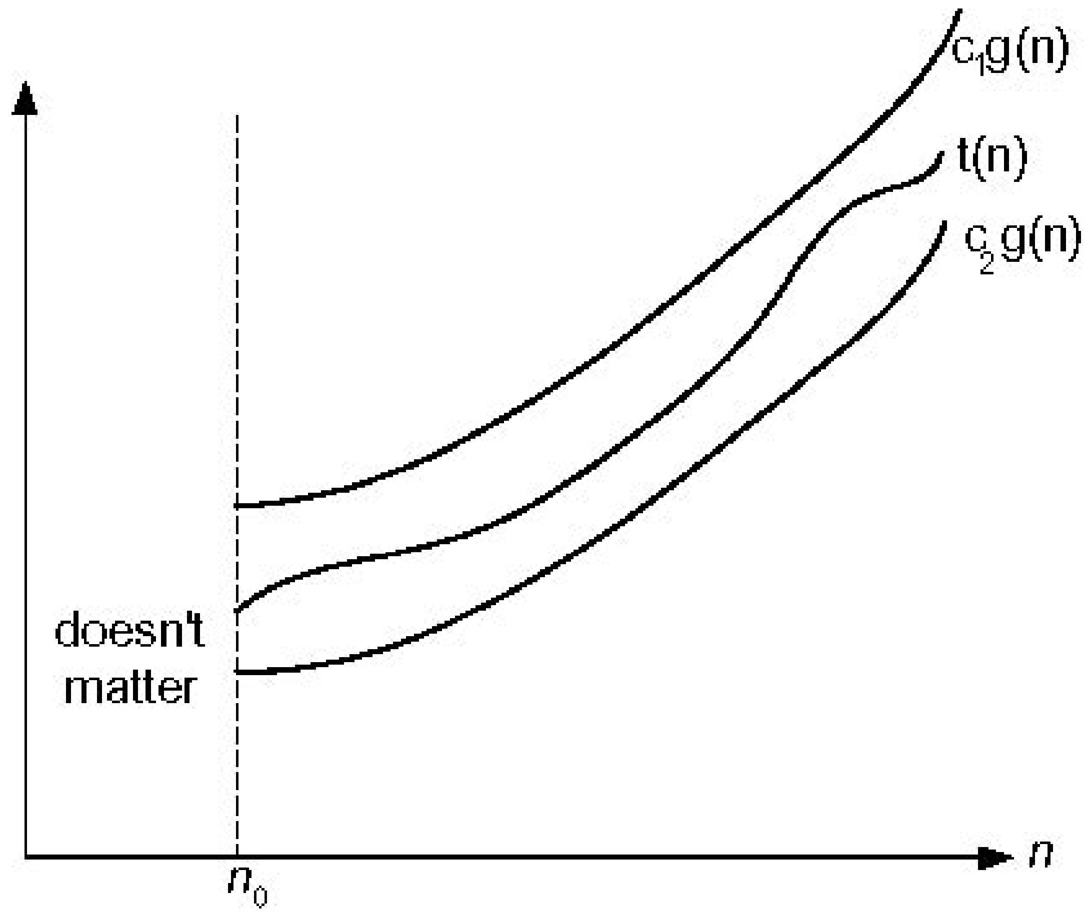


Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

Establishing order of growth using the definition

Definition: $f(n)$ **is in O(**$g(n)$**) if order of growth of** $f(n) \leq$ **order of growth of** $g(n)$ **(within constant multiple),**
i.e., there exist positive constant c **and non-negative integer** n_0 **such that**

$$f(n) \leq c\, g(n) \text{ for every } n \geq n_0$$

Examples:

- **10n is O(n^2)**

- **5n+20 is O(n)**

Some properties of asymptotic order of growth

- $f(n) \in O(f(n))$

- $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$

  Note similarity with $a \leq b$

- If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then

$$f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

Establishing order of growth using limits

$$\lim_{n\to\infty} T(n)/g(n) = \begin{cases} 0 & \text{order of growth of } T(n) < \text{ order of growth of } g(n) \\ c > 0 & \text{order of growth of } T(n) = \text{ order of growth of } g(n) \\ \infty & \text{order of growth of } T(n) > \text{ order of growth of } g(n) \end{cases}$$

Examples:

- **$10n$ vs. n^2**

- **$n(n+1)/2$ vs. n^2**

L'Hôpital's rule and Stirling's formula

L'Hôpital's rule: If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ and the derivatives f', g' exist, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Example: $\log n$ vs. n

Stirling's formula: $n! \approx (2\pi n)^{1/2} (n/e)^n$

Example: 2^n vs. $n!$

Orders of growth of some important functions

- **All logarithmic functions $\log_a n$ belong to the same class $\Theta(\log n)$ no matter what the logarithm's base $a > 1$ is**

- **All polynomials of the same degree k belong to the same class: $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 \in \Theta(n^k)$**

- **Exponential functions a^n have different orders of growth for different a's**

- **order $\log n$ < order n^α (α>0) < order a^n < order $n!$ < order n^n**

Basic asymptotic efficiency classes

1	constant
$\log n$	logarithmic
n	linear
$n \log n$	*n*-log-*n*
n^2	quadratic
n^3	cubic
2^n	exponential
$n!$	factorial

Time efficiency of nonrecursive algorithms

General Plan for Analysis

- Decide on parameter *n* indicating ***input size***
- Identify algorithm's ***basic operation***
- Determine ***worst***, ***average***, and ***best*** cases for input of size *n*
- Set up a sum for the number of times the basic operation is executed
- Simplify the sum using standard formulas and rules (see Appendix A)

Useful summation formulas and rules

$\Sigma_{l \le i \le u} 1 = 1+1+\ldots+1 = u - l + 1$

In particular, $\Sigma_{1 \le i \le u} 1 = n - 1 + 1 = n \in \Theta(n)$

$\Sigma_{1 \le i \le n}\, i = 1+2+\ldots+n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$

$\Sigma_{1 \le i \le n}\, i^2 = 1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$

$\Sigma_{0 \le i \le n}\, a^i = 1 + a + \ldots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \ne 1$

In particular, $\Sigma_{0 \le i \le n}\, 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$\Sigma(a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i$ $\quad \Sigma c a_i = c \Sigma a_i$ $\quad \Sigma_{l \le i \le u} a_i = \Sigma_{l \le i \le m} a_i + \Sigma_{m+1 \le i \le u} a_i$

Example 1: Maximum element

ALGORITHM *MaxElement*$(A[0..n-1])$

//Determines the value of the largest element in a given array
//Input: An array $A[0..n-1]$ of real numbers
//Output: The value of the largest element in A
$maxval \leftarrow A[0]$
for $i \leftarrow 1$ **to** $n-1$ **do**
    if $A[i] > maxval$
        $maxval \leftarrow A[i]$
return $maxval$

Example 1: Maximum element (Con.)

- The basic operation is a comparison of two numbers.
- C(n) is the number of times the comparison is executed.

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 = \theta(n)$$

Example 2: Element uniqueness problem

ALGORITHM *UniqueElements*$(A[0..n-1])$

//Determines whether all the elements in a given array are distinct
//Input: An array $A[0..n-1]$
//Output: Returns "true" if all the elements in A are distinct
//   and "false" otherwise

for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
        if $A[i] = A[j]$ **return false**
return true

Example 2: (Cont.)

C_{worst} is the largest number of comparison

$$C_{worst} = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1$$

$$= \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1]$$

$$= \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

$$= [(n\text{-}1) \sum_{i=0}^{n-2} 1] - [\frac{(n-2)(n-1)}{2}]$$

$$= (n\text{-}1)^2 - \frac{(n-2)(n-1)}{2}$$

$$= \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 = \theta(n^2)$$

Example 3: Matrix multiplication

ALGORITHM *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

//Multiplies two n-by-n matrices by the definition-based algorithm
//Input: Two n-by-n matrices A and B
//Output: Matrix $C = AB$
for $i \leftarrow 0$ **to** $n-1$ **do**
    for $j \leftarrow 0$ **to** $n-1$ **do**
        $C[i, j] \leftarrow 0.0$
        for $k \leftarrow 0$ **to** $n-1$ **do**
            $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$
return C

Example 4: Gaussian elimination

Algorithm *GaussianElimination*($A[0..n-1,0..n]$)
//Implements Gaussian elimination of an n-by-(n+1) matrix A
for $i \leftarrow 0$ **to** $n - 2$ **do**
    for $j \leftarrow i + 1$ **to** $n - 1$ **do**
        for $k \leftarrow i$ **to** n **do**
            $A[j,k] \leftarrow A[j,k] - A[i,k] * A[j,i] / A[i,i]$

Find the efficiency class and a constant factor improvement.

Example 5: Counting binary digits

ALGORITHM *Binary*(n)

//Input: A positive decimal integer n
//Output: The number of binary digits in n's binary representation
$count \leftarrow 1$
while $n > 1$ **do**
  $count \leftarrow count + 1$
  $n \leftarrow \lfloor n/2 \rfloor$
return $count$

It cannot be investigated the way the previous examples are.

Plan for Analysis of Recursive Algorithms

- **Decide on a parameter indicating an input's size.**
- **Identify the algorithm's basic operation.**
- **Check whether the number of times the basic op. is executed may vary on different inputs of the same size. (If it may, the worst, average, and best cases must be investigated separately.)**
- **Set up a recurrence relation with an appropriate initial condition expressing the number of times the basic op. is executed.**
- **Solve the recurrence (or, at the very least, establish its solution's order of growth) by backward substitutions or another method.**

Example 1: Recursive evaluation of n!

Definition: $n! = 1 * 2 * \ldots *(n-1) * n$ for $n \geq 1$ and $0! = 1$

Recursive definition of n!: $F(n) = F(n-1) * n$ for $n \geq 1$ and
$F(0) = 1$

ALGORITHM $F(n)$
//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

Size:
Basic operation:
Recurrence relation:

Solving the recurrence for M(*n*)

$M(n) = M(n-1) + 1, \quad M(0) = 0$

$M(n) = M(n-1) + 1$

$= [M(n-2) + 1] + 1$ because $M(n-1) = M(n-2) + 1$

$= M(n-2) + 2$

$= [M(n-3) + 1] + 2$

$= M(n-3) + 3$

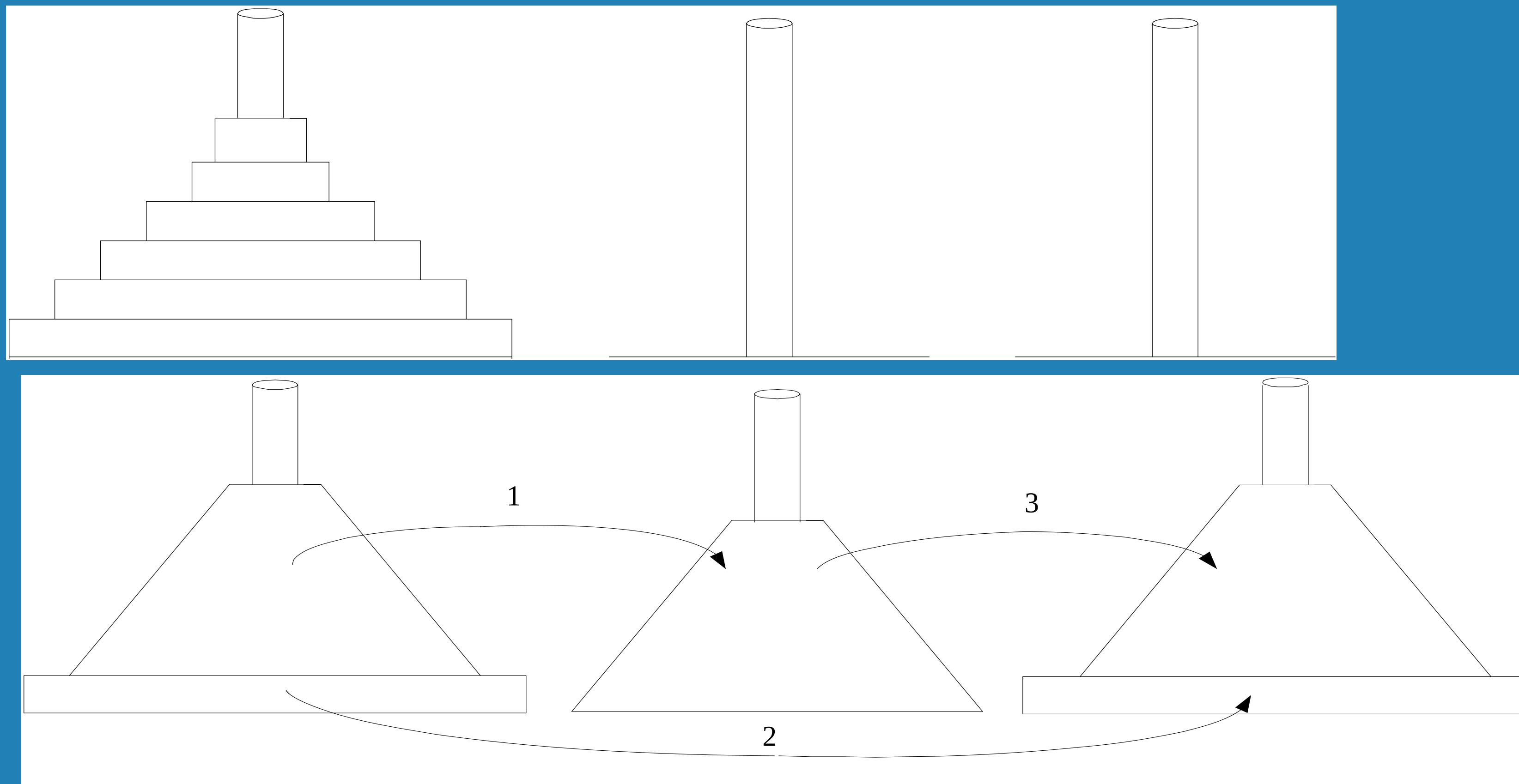
...

$= M(n-i) + i$

$= M(n-n) + n$ when $i = n$.

$= M(0) + n = n$.

Example 2: The Tower of Hanoi Puzzle

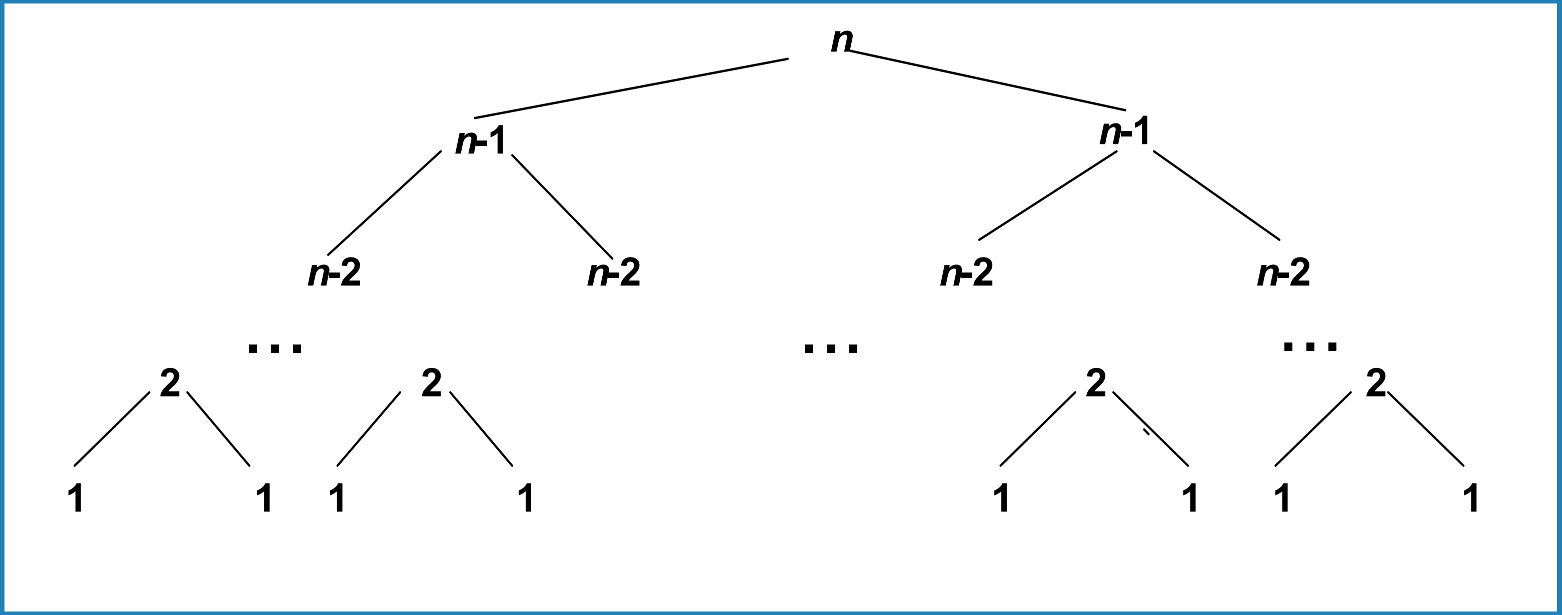


Recurrence for number of moves:

Solving recurrence for number of moves

$M(n) = 2M(n-1) + 1, \quad M(1) = 1$

Tree of calls for the Tower of Hanoi Puzzle



Example 3: Counting #bits

ALGORITHM *BinRec(n)*

//Input: A positive decimal integer *n*
//Output: The number of binary digits in *n*'s binary representation
if $n = 1$ **return** 1
else return $BinRec(\lfloor n/2 \rfloor) + 1$

The number of addition A(n) is given by

$A(n) = A(\lfloor n/2 \rfloor) + 1$ for $n > 1$
Initial condition $A(1) = 0$.
Consider n is a power of 2, i.e., $n = 2^k$, for some k.
$A(2^k) = A(2^{k-1}) + 1$ for $k > 0$
$A(2^0) = A(1) = 0$.

Solving A(n)

Solution:

$$
\begin{aligned}
A(2^k) &= A(2^{k-1}) + 1 \\
&= [A(2^{k-2} + 1] + 1 \\
&= A(2^{k-2}) + 2 \\
&\ldots \\
&= A(2^{k-i}) + i \\
&= A(2^0) + k \quad \text{for } i=k. \\
&= k.
\end{aligned}
$$

$n = 2^k$. Hence $k = \log_2 n$.

So, $A(n) = \log_2 n = \theta(\log n)$

Fibonacci numbers

The Fibonacci numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, …

The Fibonacci recurrence:

$$F(n) = F(n-1) + F(n-2)$$

$$F(0) = 0$$

$$F(1) = 1$$

General 2nd order linear homogeneous recurrence with constant coefficients:

$$aX(n) + bX(n-1) + cX(n-2) = 0$$

Solving $aX(n) + bX(n-1) + cX(n-2) = 0$

- Set up the characteristic equation (quadratic)

$$ar^2 + br + c = 0$$

- Solve to obtain roots r_1 and r_2

- General solution to the recurrence

  if r_1 and r_2 are two distinct real roots: $X(n) = \alpha r_1^{\,n} + \beta r_2^{\,n}$

  if $r_1 = r_2 = r$ are two equal real roots: $X(n) = \alpha r^n + \beta n r^n$

- Particular solution can be found by using initial conditions

Application to the Fibonacci numbers

F(n) = F(n-1) + F(n-2) or F(n) - F(n-1) - F(n-2) = 0

Characteristic equation:

Roots of the characteristic equation:

General solution to the recurrence:

Particular solution for F(0) =0, F(1)=1:

Computing Fibonacci numbers

1. **Definition-based recursive algorithm**

2. **Nonrecursive definition-based algorithm**

3. **Explicit formula algorithm**

4. **Logarithmic algorithm based on formula:**

$$\begin{pmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

for $n \geq 1$, assuming an efficient way of computing matrix powers.